LECTURE 1, 18.155, 5 SEPTEMBER 2013

Outline:

(1) Distributions
    Structure theorems
    Homogeneous distributions
    Convolution
    Homogeneous distributions
(2) Constant coefficient differential operators
    Elliptic operators and regularity
    Fundamental solutions
    D'Alembertian and wavefront set
(3) Operators
    Spectral theorem
    Fredholm operators
    Homogeneous elliptic operators

And more as time permits!

Today:- Skimming over Chapter 1 and then into Chapter 3 of current notes to §2.

- Riesz' Representation for $L^2(\mathbb{R}^n)$.

  Why we need the Lebesgue integral and especially $L^2(\mathbb{R}^n)$. For at least two reasons – but basically the fact that it is intimately connected to the integral pairing of functions

$$(f, g) = \int_{\mathbb{R}^n} fg.$$

  [Or the sesquilinear version of this.] One way to think about the sort of problem we are interested in, is that we might somehow construct a continuous linear function on $L^2(\mathbb{R}^n)$ and then use Riesz' theorem to see that it is defined by pairing with some function $g \in L^2(\mathbb{R}^n)$. Then in an application we would very much like to know how smooth g actually is – probably it is much 'better' than just an L^2 function. This way of thinking leads to the 'tower of Sobolev spaces' that we will want to understand:

$$\begin{array}{c}
H^\infty \subset \mathcal{C}^\infty(\mathbb{R}^n) \\
\downarrow \\
\cdots \\
\downarrow \\
H^N(\mathbb{R}^n) \\
\downarrow \\
\cdots \\
\downarrow \\
H^1(\mathbb{R}^n) \\
\downarrow \\
L^2(\mathbb{R}^n) = H^0(\mathbb{R}^n)
\end{array}$$

Here the successive inclusions are dense and the exponent N in $H^N(\mathbb{R}^n)$ represents the number of 'L^2 derivatives' a function has.

- In fact I will start off by talking about distributions – which is what you get by taking the duality idea seriously. Initially at least I will talk about tempered distributions but the idea is that $H^N(\mathbb{R}^n)$ is 'included' as a dense subspace of $L^2(\mathbb{R}^n)$. So we might want to think of any continously linear functional on $H^N(\mathbb{R}^n)$ as somehow being a 'badly behaved functional on $L^2(\mathbb{R}^n)$.' This was what Dirac did – he thought such functionals should be considered as 'improper functions' but later these were called generalized functions or distributions. Really what happens is that the dual of $H^N(\mathbb{R}^n)$ can be realized a space of objects into which $L^2(\mathbb{R}^n)$ includes and we can continue the tower downwards.

$$\begin{array}{c} L^2(\mathbb{R}^n) = H^0(\mathbb{R}^n) \dots \\ \big\downarrow \\ H^{-1}(\mathbb{R}^n) \\ \big\downarrow \\ H^{-N}(\mathbb{R}^n) \\ \big\downarrow \\ \dots \\ \big\downarrow \\ H^{-\infty} \subset \mathcal{S}'(\mathbb{R}^n) = \text{tempered distributions.} \end{array}$$

Duality represents reflection in the tower around the middle space, $L^2(\mathbb{R}^n)$ which is self-dual.

- Thus we will construct a 'big space' in which most reasonable things are contained and work out how to show that they are in smaller, 'better' spaces. The big space however is just the dual of a small space which we discuss first, and indeed the properties of the big space typically correspond to some (other) property of the small space.
- I will quickly describe Banach spaces of bounded continuous functions and of functions with bounded and continuous partial derivatives up to some finite order N. Then discuss polynomially weighted spaces. All this to lead up the basic definition of the *space of Schwartz test functions:*

$$\mathcal{S}(\mathbb{R}^n) = \{f : \mathbb{R}^n \longrightarrow \mathbb{C};\ \text{all partial derivatives } \partial^\alpha f \text{ exist and} \\ \|f\|_{\alpha,\beta} = \sup_x |x^\beta \partial^\alpha f(x)| < \infty\ \forall\ \alpha,\ \beta \in \mathbb{N}_0^n\}.$$

Here multi-index notation for monomials and derivatives is used: x^β and ∂^α.

- The Gaussian is in the Schwartz space of test functions, $\exp(-|x|^2) \in \mathcal{S}(\mathbb{R}^n)$. Here it follows by induction that

$$\partial^\alpha \exp(-|x|^2) = P_\alpha(x) \exp(-|x|^2) \tag{1}$$

where $P_\alpha(x)$ is a polynomial of degree $|\alpha|$ (it is an unormalized Hermite polynomial).

- Finishing today, or more likely next Tuesday, with the fact that $\mathcal{S}(\mathbb{R}^n)$ is a complete metric space and hence continuous functions on it are well-defined.