CHAPTER 4

Elliptic Regularity

Includes some corrections noted by Tim Nguyen and corrections by, and some suggestions from, Jacob Bernstein.

1. Constant coefficient operators

A linear, constant coefficient differential operator can be thought of as a map

$$P(D) : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \text{ of the form } P(D)u(z) = \sum_{|\alpha| \le m} c_\alpha D^\alpha u(z),$$
$$D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n},\ D_j = \frac{1}{i}\frac{\partial}{\partial z_j}, \tag{1.1}$$

but it also acts on various other spaces. So, really it is just a polynomial $P(\zeta)$ in n variables. This 'characteristic polynomial' has the property that

$$\mathcal{F}(P(D)u)(\zeta) = P(\zeta)\mathcal{F}u(\zeta), \tag{1.2}$$

which you may think of as a little square

$$\begin{array}{ccc} \mathcal{S}(\mathbb{R}^n) & \xrightarrow{P(D)} & \mathcal{S}(\mathbb{R}^n) \\ {\scriptstyle\mathcal{F}}\updownarrow & & \updownarrow{\scriptstyle\mathcal{F}} \\ \mathcal{S}(\mathbb{R}^n) & \xrightarrow[P\times]{} & \mathcal{S}(\mathbb{R}^n) \end{array} \tag{1.3}$$

and this is why the Fourier tranform is especially useful. However, this still does not solve the important questions directly.

Question 1.1. $P(D)$ is always injective as a map (1.1) but is usually not surjective. When is it surjective? If $\Omega \subset \mathbb{R}^n$ is a non-empty open set then

$$P(D) : \mathcal{C}^\infty(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega) \tag{1.4}$$

is never injective (unless $P(\zeta)$ is constnat), for which polynomials is it surjective?

The first three points are relatively easy. As a map (1.1) $P(D)$ is injective since if $P(D)u = 0$ then by (1.2), $P(\zeta)\mathcal{F}u(\zeta) = 0$ on $\mathbb{R}^n$. However, a zero set, in $\mathbb{R}^n$, of a non-trivial polynomial alwasys has empty interior, i.e. the set where it is non-zero is dense, so $\mathcal{F}u(\zeta) = 0$ on $\mathbb{R}^n$ (by continuity) and hence $u = 0$ by the invertibility of the Fourier transform. So (1.1) is injective (of course excepting the case that P is the zero polynomial). When is it surjective? That is, when can every $f \in \mathcal{S}(\mathbb{R}^n)$ be written as $P(D)u$ with $u \in \mathcal{S}(\mathbb{R}^n)$? Taking the Fourier transform again, this is the same as asking when every $g \in \mathcal{S}(\mathbb{R}^n)$ can be written in the form $P(\zeta)v(\zeta)$ with $v \in \mathcal{S}(\mathbb{R}^n)$. If $P(\zeta)$ has a zero in $\mathbb{R}^n$ then this is not possible, since $P(\zeta)v(\zeta)$ always vanishes at such a point. It is a little trickier to see the converse, that $P(\zeta) \neq 0$ on $\mathbb{R}^n$ implies that $P(D)$ in (1.1) is surjective. Why is this not obvious? Because we need to show that $v(\zeta) = g(\zeta)/P(\zeta) \in \mathcal{S}(\mathbb{R}^n)$ whenever $g \in \mathcal{S}(\mathbb{R}^n)$. Certainly, $v \in \mathcal{C}^\infty(\mathbb{R}^n)$ but we need to show that the derivatives decay rapidly at infinity. To do this we need to get an estimate on the rate of decay of a non-vanishing polynomial

Lemma 1.1. *If P is a polynomial such that $P(\zeta) \neq 0$ for all $\zeta \in \mathbb{R}^n$ then there exists $C > 0$ and $a \in \mathbb{R}$ such that*

$$|P(\zeta)| \geq C(1 + |\zeta|)^a. \tag{1.5}$$

Proof. This is a form of the Tarski-Seidenberg Lemma. Stated loosely, a semi-algebraic function has power-law bounds. Thus

$$F(R) = \inf\{|P(\zeta)|; |\zeta| \leq R\} \tag{1.6}$$

is semi-algebraic and non-vanishing so must satisfy $F(R) \geq c(1 + R)^a$ for some $c > 0$ and a (possibly negative). This gives the desired bound. Is there an elementary proof? □

Thirdly the non-injectivity in (1.4) is obvious for the opposite reason. Namely for any non-constant polynomial there exists $\zeta \in \mathbb{C}^n$ such that $P(\zeta) = 0$. Since

$$P(D)e^{i\zeta\cdot z} = P(\zeta)e^{i\zeta\cdot z} \tag{1.7}$$

such a zero gives rise to a non-trivial element of the null space of (1.4). You can find an extensive discussion of the density of these sort of 'exponential' solutions (with polynomial factors) in all solutions in Hörmander's book [**4**].

What about the surjectivity of (1.4)? It is not always surjective unless Ω is *convex* but there are decent answers, to find them you should look under P-convexity in [**4**]. If $P(\zeta)$ is elliptic then (1.4) is surjective for every open Ω; maybe I will prove this later although it is not a result of great utility.

2. Constant coefficient elliptic operators

To discuss elliptic regularity, let me recall that any constant coefficient differential operator of order m defines a continuous linear map

$$P(D) : H^{s+m}(\mathbb{R}^n) \longmapsto H^s(\mathbb{R}^n). \tag{2.1}$$

Provided P is not the zero polynomial this map is *always* injective. This follows as in the discussion above for $\mathcal{S}(\mathbb{R}^n)$. Namely, if $u \in H^{s+m}(\mathbb{R}^n)$ then, by definition, $\hat{u} \in L^2_{\text{loc}}(\mathbb{R}^n)$ and if $Pu = 0$ then $P(\zeta)\hat{u}(\zeta) = 0$ off a set of measure zero. Since $P(\zeta) \neq 0$ on an open dense set it follows that $\hat{u} = 0$ off a set of measure zero and so $u = 0$ as a distribution.

As a map (2.1), $P(D)$ is is seldom surjective. It is said to be elliptic (either as a polynomial or as a differential operator) if it is of order m and there is a constant $c > 0$ such that

$$|P(\zeta)| \geq c(1 + |\zeta|)^m \text{ in } \{\zeta \in \mathbb{R}^n; |\zeta| > 1/c\}. \tag{2.2}$$

Proposition 2.1. *As a map* (2.1)*, for a given* s*,* $P(D)$ *is surjective if and only if* P *is elliptic and* $P(\zeta) \neq 0$ *on* $\mathbb{R}^n$ *and then it is a topological isomorphism for every* s*.*

Proof. Since the Sobolev spaces are defined as the Fourier transforms of the weighted L^2 spaces, that is

$$f \in H^t(\mathbb{R}^n) \Longleftrightarrow (1 + |\zeta|^2)^{t/2}\hat{f} \in L^2(\mathbb{R}^n) \tag{2.3}$$

the sufficiency of these conditions is fairly clear. Namely the combination of ellipticity, as in (2.2), and the condition that $P(\zeta) \neq 0$ for $\zeta \in \mathbb{R}^n$ means that

$$|P(\zeta)| \geq c(1 + |\zeta|^2)^{m/2},\ c > 0,\ \zeta \in \mathbb{R}^n. \tag{2.4}$$

From this it follows that $P(\zeta)$ is bounded above and below by multiples of $(1 + |\zeta|^2)^{m/2}$ and so maps the weighted L^2 spaces into each other

$$\times P(\zeta) : H^{0,s+m}(\mathbb{R}^n) \longrightarrow H^{0,s}(\mathbb{R}^n),\ H^{0,s} = \{u \in L^2_{\text{loc}}(\mathbb{R}^n); \langle\zeta\rangle^s u(\zeta) \in L^2(\mathbb{R}^n)\}, \tag{2.5}$$

giving an isomorphism (2.1) after Fourier transform.

The necessity follows either by direct construction or else by use of the closed graph theorem. If $P(D)$ is surjective then multiplication by $P(\zeta)$ must be an isomorphism between the corresponding weighted space $H^{0,s}(\mathbb{R}^n)$ and $H^{0,s+m}(\mathbb{R}^n)$. By the density of functions supported off the zero set of P the norm of the inverse can be seen to be the inverse of

$$\inf_{\zeta \in \mathbb{R}^n} |P(\zeta)|\langle\zeta\rangle^{-m} \tag{2.6}$$

which proves ellipticity. □

Ellipticity is reasonably common in appliactions, but the condition that the characteristic polynomial not vanish at all is frequently not satisfied. In fact one of the questions I want to get to in this course – even though we are interested in variable coefficient operators – is improving (2.1) (by changing the Sobolev spaces) to get an isomorphism at least for homogeneous elliptic operators (which do not satisfy the second condition in Proposition 2.1 because they vanish at the origin). One reason for this is that we want it for monopoles.

Note that ellipticity itself is a condition on the principal part of the polynomial.

LEMMA 2.2. *A polynomial* $P(\zeta) = \sum\limits_{\alpha| \le m} c_\alpha \zeta^\alpha$ *of degree* m *is elliptic if and only if its leading part*

$$P_m(\zeta) = \sum_{|\alpha|=m} c_\alpha \zeta^\alpha \neq 0 \text{ on } \mathbb{R}^n \setminus \{0\}. \tag{2.7}$$

PROOF. Since the principal part is homogeneous of degree m the requirement (2.7) is equivalent to

$$|P_m(\zeta)| \ge c|\zeta|^m, \ c = \inf_{|\zeta|=1} |P(\zeta)| > 0. \tag{2.8}$$

Thus, (2.2) follows from this, since

$$|P(\zeta)| \ge |P_m(\zeta)| - |P'(\zeta)| \ge c|\zeta|^m - C|\zeta|^{m-1} - C \ge \frac{c}{2}|\zeta|^m \text{ if } |\zeta| > C', \tag{2.9}$$

$P' = P - M_m$ being of degree at most $m-1$. Conversely, ellipticity in the sense of (2.2) implies that

$$|P_m(\zeta)| \ge |P(\zeta)| - |P'(\zeta)| \ge c|\zeta|^m - C|\zeta|^{m-1} - C > 0 \text{ in } |\zeta| > C' \tag{2.10}$$

and so $P_m(\zeta) \neq 0$ for $\zeta \in \mathbb{R}^n \setminus \{0\}$ by homogeneity. □

Let me next recall *elliptic regularity* for constant coefficient operators. Since this is a local issue, I first want to recall the local versions of the Sobolev spaces discussed in Chapter 3

DEFINITION 2.3. *If* $\Omega \subset \mathbb{R}^n$ *is an open set then*

$$H^s_{\text{loc}}(\Omega) = \left\{ u \in \mathcal{C}^{-\infty}(\Omega); \phi u \in H^s(\mathbb{R}^n) \ \forall \ \phi \in \mathcal{C}^\infty_c(\Omega) \right\}. \tag{2.11}$$

Again you need to know what $\mathcal{C}^{-\infty}(\Omega)$ is (it is the dual of $\mathcal{C}^\infty_c(\Omega)$) and that multiplication by $\phi \in \mathcal{C}^\infty_c(\Omega)$ defines a linear continuous map from $\mathcal{C}^{-\infty}(\mathbb{R}^n)$ to $\mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ and gives a bounded operator on $H^m(\mathbb{R}^n)$ for all m.

Proposition 2.4. *If $P(D)$ is elliptic, $u \in \mathcal{C}^{-\infty}(\Omega)$ is a distribution on an open set and $P(D)u \in H^s_{\text{loc}}(\Omega)$ then $u \in H^{s+m}_{\text{loc}}(\Omega)$. Furthermore if ϕ, $\psi \in \mathcal{C}^\infty_c(\Omega)$ with $\phi = 1$ in a neighbourhood of* $\operatorname{supp}(\psi)$ *then*

$$\|\psi u\|_{s+m} \le C\|\psi P(D)u\|_s + C'\|\phi u\|_{s+m-1} \tag{2.12}$$

for any $M \in \mathbb{R}$, with C' depending only on ψ, ϕ, M and $P(D)$ and C depending only on $P(D)$ (so neither depends on u).

Although I will no prove it here, and it is not of any use below, it is worth noting that (2.12) *characterizes* the ellipticity of a differential operator with smooth coefficients.

Proof. Let me discuss this in two slightly different ways. The first, older, approach is via direct regularity estimates. The second is through the use of a parametrix; they are not really very different!

First the regularity estimates. An easy case of Proposition 2.4 arises if $u \in \mathcal{C}^{-\infty}_c(\Omega)$ has compact support to start with. Then $P(D)u$ also has compact support so in this case

$$u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n) \text{ and } P(D)u \in H^s(\mathbb{R}^n). \tag{2.13}$$

Then of course the Fourier transform works like a charm. Namely $P(D)u \in H^s(\mathbb{R}^n)$ means that

$$\langle\zeta\rangle^s P(\zeta)\hat{u}(\zeta) \in L^2(\mathbb{R}^n) \Longrightarrow \langle\zeta\rangle^{s+m}F(\zeta)\hat{u}(\zeta) \in L^2(\mathbb{R}^n),\ F(\zeta) = \langle\zeta\rangle^{-m}P(\zeta). \tag{2.14}$$

Ellipticity of $P(\zeta)$ implies that $F(\zeta)$ is bounded above and below on $|\zeta| > 1/c$ and hence can be inverted there by a bounded function. It follows that, given any $M \in \mathbb{R}$ the norm of u in $H^{s+m}(\mathbb{R}^n)$ is bounded

$$\|u\|_{s+m} \le C\|u\|_s + C'_M\|u\|_M,\ u \in \mathcal{C}^{-\infty}(\Omega), \tag{2.15}$$

where the second term is used to bound the L^2 norm of the Fourier transform in $|\zeta| \le 1/c$.

To do the general case of an open set we need to use cutoffs more seriously. We want to show that $\psi u \in H^{s+m}(\mathbb{R}^n)$ where $\psi \in \mathcal{C}^\infty_c(\Omega)$ is some fixed but arbitrary element. We can always choose some function $\phi \in \mathcal{C}^\infty_c(\Omega)$ which is equal to one in a neighbourhood of the support of ψ. Then $\phi u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ so, by the Schwartz structure theorem, $\phi u \in H^{m+t-1}(\mathbb{R}^n)$ for some (unknown) $t \in \mathbb{R}$. We will show that $\psi u \in H^{m+T}(\mathbb{R}^n)$ where T is the smaller of s and t. To see this, compute

$$P(D)(\psi u) = \psi P(D)u + \sum_{|\beta|\le m-1, |\gamma|\ge 1} c_{\beta,\gamma} D^\gamma \psi D^\beta \phi u. \tag{2.16}$$

With the final ϕu replaced by u this is just the effect of expanding out the derivatives on the product. Namely, the $\psi P(D)u$ term is when no derivative hits ψ and the other terms come from at least one derivative hitting ψ. Since $\phi = 1$ on the support of ψ we may then insert ϕ without changing the result. Thus the first term on the right in (2.16) is in $H^s(\mathbb{R}^n)$ and all terms in the sum are in $H^t(\mathbb{R}^n)$ (since at most $m-1$ derivatives are involved and $\phi u \in H^{m+t-1}(\mathbb{R}^n)$ be definition of t). Applying the simple case discussed above it follows that $\psi u \in H^{m+r}(\mathbb{R}^n)$ with r the minimum of s and t. This would result in the estimate

$$\|\psi u\|_{s+m} \leq C\|\psi P(D)u\|_s + C'\|\phi u\|_{s+m-1} \tag{2.17}$$

provided we knew that $\phi u \in H^{s+m-1}$ (since then $t = s$). Thus, initially we only have this estimate with s replaced by T where $T = \min(s,t)$. However, the only obstruction to getting the correct estimate is knowing that $\psi u \in H^{s+m-1}(\mathbb{R}^n)$.

To see this we can use a bootstrap argument. Observe that ψ can be taken to be *any* smooth function with support in the interior of the set where $\phi = 1$. We can therefore insert a chain of functions, of any finite (integer) length $k \geq s-t$, between then, with each supported in the region where the previous one is equal to 1 :

$$\operatorname{supp}(\psi) \subset \{\phi_k = 1\}^\circ \subset \operatorname{supp}(\phi_k) \subset \cdots \subset \operatorname{supp}(\phi_1) \subset \{\phi = 1\}^\circ \subset \operatorname{supp}(\phi) \tag{2.18}$$

where ψ and ϕ were our initial choices above. Then we can apply the argument above with $\psi = \phi_1$, then $\psi = \phi_2$ with ϕ replaced by ϕ_1 and so on. The initial regularity of $\phi u \in H^{t+m-1}(\mathbb{R}^n)$ for some t therefore allows us to deduce that

$$\phi_j u \in H^{m+T_j}(\mathbb{R}^n),\ T_j = \min(s, t+j-1). \tag{2.19}$$

If k is large enough then $\min(s, t+k) = s$ so we conclude that $\psi u \in H^{s+m}(\mathbb{R}^n)$ for any such ψ and that (2.17) holds. □

Although this is a perfectly adequate proof, I will now discuss the second method to get elliptic regularity; the main difference is that we think more in terms of operators and avoid the explicit iteration technique, by doing it all at once – but at the expense of a little more thought. Namely, going back to the easy case of a tempered distibution on $\mathbb{R}^n$ give the map a name:-

$$Q(D) : f \in \mathcal{S}'(\mathbb{R}^n) \longmapsto \mathcal{F}^{-1}\left(\hat{q}(\zeta)\hat{f}(\zeta)\right) \in \mathcal{S}'(\mathbb{R}^n),\ \hat{q}(\zeta) = \frac{1-\chi(\zeta)}{P(\zeta)}. \tag{2.20}$$

Here $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is chosen to be equal to one on the set $|\zeta| \leq \frac{1}{c} + 1$ corresponding to the ellipticity estimate (2.2). Thus $\hat{q}(\zeta) \in \mathcal{C}^\infty(\mathbb{R}^n)$ is bounded and in fact

$$|D^\alpha_\zeta \hat{q}(\zeta)| \le C_\alpha (1+|\zeta|)^{-m-|\alpha|} \ \forall \ \alpha. \tag{2.21}$$

This has a straightforward proof by induction. Namely, these estimates are trivial on any compact set, where the function is smooth, so we need only consider the region where $\chi(\zeta) = 0$. The inductive statement is that for some polynomials H_α,

$$D^\alpha_\zeta \frac{1}{P(\zeta)} = \frac{H_\alpha(\zeta)}{(P(\zeta))^{|\alpha|+1}},\ \deg(H_\alpha) \le (m-1)|\alpha|. \tag{2.22}$$

From this (2.21) follows. Prove (2.22) itself by differentiating one more time and reorganizing the result.

So, in view of the estimate with $\alpha = 0$ in (2.21),

$$Q(D) : H^s(\mathbb{R}^n) \longrightarrow H^{s+m}(\mathbb{R}^n) \tag{2.23}$$

is continuous for each s and it is also an essential inverse of $P(D)$ in the sense that as operators on $\mathcal{S}'(\mathbb{R}^n)$

(2.24)

$$Q(D)P(D) = P(D)Q(D) = \operatorname{Id} - E,\ E : H^s(\mathbb{R}^n) \longrightarrow H^\infty(\mathbb{R}^n) \ \forall \ s \in \mathbb{R}.$$

Namely, E is Fourier multiplication by a smooth function of compact support (namely $1 - \hat{q}(\zeta)P(\zeta)$. So, in the global case of $\mathbb{R}^n$, we get elliptic regularity by applying $Q(D)$ to both sides of the equation $P(D)u = f$ to find

$$f \in H^s(\mathbb{R}^n) \Longrightarrow u = Eu + Qf \in H^{s+m}(\mathbb{R}^n). \tag{2.25}$$

This also gives the esimate (2.15) where the second term comes from the continuity of E.

The idea then, is to do the same thing for $P(D)$ acting on functions on the open set Ω; this argument will subsequently be generalized to variable coefficient operators. The problem is that $Q(D)$ does not act on functions (or chapterdistributions) defined just on Ω, they need to be defined on the whole of $\mathbb{R}^n$ and to be tempered before the the Fourier transform can be applied and then multiplied by $\hat{q}(\zeta)$ to define $Q(D)f$.

Now, $Q(D)$ is a convolution operator. Namely, rewriting (2.20)

$$Q(D)f = Qf = q * f,\ q \in \mathcal{S}'(\mathbb{R}^n),\ \hat{q} = \frac{1-\chi(\zeta)}{P(\zeta)}. \tag{2.26}$$

This in fact is exactly what (2.20) means, since $\mathcal{F}(q * f) = \hat{q}\hat{f}$. We can write out convolution by a smooth function (which q is not, but let's not quibble) as an integral

$$q * f(\zeta) = \int_{\mathbb{R}^n} q(z - z')f(z')dz'. \tag{2.27}$$

Restating the problem, (2.27) is an integral (really a distributional pairing) over the whole of $\mathbb{R}^n$ not the subset Ω. In essence the cutoff argument above inserts a cutoff ϕ in front of f (really of course in front of u but not to worry). So, let's think about inserting a cutoff into (2.27), replacing it by

$$Q_\psi f(\zeta) = \int_{\mathbb{R}^n} q(z-z')\chi(z,z')f(z')dz'. \tag{2.28}$$

Here we will take $\chi \in \mathcal{C}^\infty(\Omega^2)$. To get the integrand to have compact support in Ω for each $z \in \Omega$ we want to arrange that the projection onto the second variable, z'

$$\pi_L : \Omega \times \Omega \supset \operatorname{supp}(\chi) \longrightarrow \Omega \tag{2.29}$$

should be proper, meaning that the inverse image of a compact subset $K \subset \Omega$, namely $(\Omega \times K) \cap \operatorname{supp}(\chi)$, should be compact in Ω.

Let me strengthen the condition on the support of χ by making it more two-sided and demand that $\chi \in \mathcal{C}^\infty(\Omega^2)$ have proper support in the following sense:

(2.30)
If $K \subset \Omega$ then $\pi_R\left((\Omega \times K) \cap \operatorname{supp}(\chi)\right) \cup \pi_L\left((L \times \Omega) \cap \operatorname{supp}(\chi)\right) \Subset \Omega$.

Here π_L, $\pi_R : \Omega^2 \longrightarrow \Omega$ are the two projections, onto left and right factors. This condition means that if we multiply the integrand in (2.28) on the left by $\phi(z)$, $\phi \in \mathcal{C}_c^\infty(\Omega)$ then the integrand has compact support in z' as well – and so should exist at least as a distributional pairing. The second property we want of χ is that it should not change the properties of q as a convolution operator too much. This reduces to

$$\chi = 1 \text{ in a neighbourhood of } \operatorname{Diag} = \{(z,z); z \in \Omega\} \subset \Omega^2 \tag{2.31}$$

although we could get away with the weaker condition that

$$\chi \equiv 1 \text{ in Taylor series at } \operatorname{Diag}. \tag{2.32}$$

Before discussing why these conditions help us, let me just check that it is possible to find such a χ. This follows easily from the existence of a partition of unity in Ω as follows. It is possible to find functions $\phi_i \in \mathcal{C}_c^\infty(\Omega)$, $i \in \mathbb{N}$, which have locally finite supports (i.e. any compact subset of Ω only meets the supports of a finite number of the ϕ_i,) such that $\sum\limits_i \phi_i(z) = 1$ in Ω and also so there exist functions $\phi_i' \in \mathcal{C}_c^\infty(\Omega)$, also with locally finite supports in the same sense and such that $\phi_i' = 1$ on a neighborhood of the support of ϕ_i. The existence of such functions is a standard result, or if you prefer, an exercise.

Accepting that such functions exists, consider

$$\chi(z,z') = \sum_i \phi_i(z)\phi_i'(z'). \tag{2.33}$$

Any compact subset of Ω^2 is contained in a compact set of the form $K \times K$ and hence meets the supports of only a finite number of terms in (2.33). Thus the sum is locally finite and hence $\chi \in \mathcal{C}^\infty(\Omega^2)$. Moreover, its support has the property (2.30). Clearly, by the assumption that $\phi_i' = 1$ on the support of ϕ_i and that the latter form a partition of unity, $\chi(z,z) = 1$. In fact $\chi(z,z') = 1$ in a neighborhood of the diagonal since each z has a neighborhood N such that $z' \in N$, $\phi_i(z) \neq 0$ implies $\phi_i'(z') = 1$. Thus we have shown that such a cutoff function χ exists.

Now, why do we want (2.31)? This arises because of the following 'pseudolocal' property of the kernel q.

Lemma 2.5. *Any distribution q defined as the inverse Fourier transform of a function satisfying* (2.21) *has the property*

$$\operatorname{singsupp}(q) \subset \{0\} \tag{2.34}$$

Proof. This follows directly from (2.21) and the properties of the Fourier transform. Indeed these estimates show that

$$z^\alpha q(z) \in \mathcal{C}^N(\mathbb{R}^n) \text{ if } |\alpha| > n + N \tag{2.35}$$

since this is enough to show that the Fourier transform, $(i\partial_\zeta)^\alpha \hat{q}$, is L^1. At every point of $\mathbb{R}^n$, other than 0, one of the z_j is non-zero and so, taking $z^\alpha = z_j^k$, (2.35) shows that $q(z)$ is in C^N in $\mathbb{R}^n \setminus \{0\}$ for all N, i.e. (2.34) holds. □

Thus the distribution $q(z - z')$ is only singular at the diagonal. It follows that different choices of χ with the properties listed above lead to kernels in (2.28) which differ by smooth functions in Ω^2 with proper supports.

Lemma 2.6. *A properly supported smoothing operator, which is by defninition given by an integral operator*

$$Ef(z) = \int_\Omega E(z,z')f(z')dz' \tag{2.36}$$

where $E \in \mathcal{C}^\infty(\Omega^2)$ has proper support (so both maps

$$\pi_L,\ \pi_R : \operatorname{supp}(E) \longrightarrow \Omega \tag{2.37}$$

are proper), defines continuous operators

$$E : \mathcal{C}^{-\infty}(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega),\ \mathcal{C}_c^{-\infty}(\Omega) \longrightarrow \mathcal{C}_c^\infty(\Omega) \tag{2.38}$$

and has an adjoint of the same type.

See the discussion in Chapter 3.

Proposition 2.7. *If $P(D)$ is an elliptic operator with constant coefficients then the kernel in* (2.28) *defines an operator* $Q_\Omega : \mathcal{C}^{-\infty}(\Omega) \longrightarrow \mathcal{C}^{-\infty}(\Omega)$ *which maps* $H^s_{\text{loc}}(\Omega)$ *to* $H^{s+m}_{\text{loc}}(\Omega)$ *for each* $s \in \mathbb{R}$ *and gives a 2-sided parametrix for* $P(D)$ *in* Ω :

$$P(D)Q_\Omega = \operatorname{Id} - R,\ \ Q_\Omega P(D) = \operatorname{Id} - R' \tag{2.39}$$

where R *and* R' *are properly supported smoothing operators.*

Proof. We have already seen that changing χ in (2.28) changes Q_Ω by a smoothing operator; such a change will just change R and R' in (2.39) to different properly supported smoothing operators. So, we can use the explicit choice for χ made in (2.33) in terms of a partition of unity. Thus, multiplying on the left by some $\mu \in \mathcal{C}^\infty_c(\Omega)$ the sum becomes finite and

$$\mu Q_\Omega f = \sum_j \mu\psi_j q * (\psi'_j f). \tag{2.40}$$

It follows that Q_Ω acts on $\mathcal{C}^{-\infty}(\Omega)$ and, from the properties of q it maps $H^s_{\text{loc}}(\mathbb{R}^n)$ to $H^{s+m}_{\text{loc}}(\mathbb{R}^n)$ for any s. To check (2.39) we may apply $P(D)$ to (2.40) and consider a region where $\mu = 1$. Since $P(D)q = \delta_0 - \tilde{R}$ where $\tilde{R} \in \mathcal{S}(\mathbb{R}^n)$, $P(D)Q_\Omega f = \operatorname{Id} - R$ where additional 'error terms' arise from any differentiation of ϕ_j. All such terms have smooth kernels (since $\phi'_j = 1$ on the support of ϕ_j and $q(z - z')$ is smooth outside the diagonal) and are properly supported. The second identity in (2.39) comes from the same computation for the adjoints of $P(D)$ and Q_Ω. □

3. Interior elliptic estimates

Next we proceed to prove the same type of regularity and estimates, (2.17), for elliptic differential operators with variable coefficients. Thus consider

$$P(z, D) = \sum_{|\alpha| \le m} p_\alpha(z) D^\alpha,\ \ p_\alpha \in \mathcal{C}^\infty(\Omega). \tag{3.1}$$

We now assume ellipticity, of fixed order m, for the polynomial $P(z, \zeta)$ for each $z \in \Omega$. This is the same thing as ellipticity for the principal part, i.e. the condition for each compact subset of Ω

$$|\sum_{|\alpha|=m} p_\alpha(z)\zeta^\alpha| \ge C(K)|\zeta|^m,\ \ z \in K \Subset \Omega, C(K) > 0. \tag{3.2}$$

Since the coefficients are smooth this and $\mathcal{C}^\infty(\Omega)$ is a multiplier on $H^s_{\text{loc}}(\Omega)$ such a differential operator (elliptic or not) gives continuous linear maps

$$P(z,D):H^{s+m}_{\mathrm{loc}}(\Omega)\longrightarrow H^s_{\mathrm{loc}}(\Omega),\ \forall\, s\in\mathbb{R},\ P(z,D):\mathcal{C}^\infty(\Omega)\longrightarrow\mathcal{C}^\infty(\Omega). \tag{3.3}$$

Now, we arrived at the estimate (2.12) in the constant coefficient case by iteration from the case $M=s+m-1$ (by nesting cutoff functions). Pick a point $\bar z\in\Omega$. In a small ball around $\bar z$ the coefficients are 'almost constant'. In fact, by Taylor's theorem,

$$P(z,\zeta)=P(\bar z,\zeta)+Q(z,\zeta),\ \ Q(z,\zeta)=\sum_j (z-\bar z)_j P_j(z,\bar z,\zeta) \tag{3.4}$$

where the P_j are also polynomials of degree m in ζ and smooth in z in the ball (and in $\bar z$.) We can apply the estimate (2.12) for $P(\bar z,D)$ and $s=0$ to find

$$\|\psi u\|_m\le C\|\psi\left(P(z,D)u-Q(z,D)\right)u\|_0+C'\|\phi u\|_{m-1}. \tag{3.5}$$

Because the coefficients are small

$$\begin{aligned}\|\psi Q(z,D)u\|_0&\le\sum_{j,|\alpha|\le m}\|(z-\bar z)_j r_{j,\alpha}D^\alpha\psi u\|_0+C'\|\phi u\|_{m-1}\\&\le\delta C\|\psi u\|_m+C'\|\phi u\|_{m-1}.\end{aligned} \tag{3.6}$$

What we would like to say next is that we can choose δ so small that $\delta C<\frac12$ and so inserting (3.6) into (3.5) we would get

$$\begin{aligned}\|\psi u\|_m&\le C\|\psi P(z,D)u\|_0+C\|\psi Q(z,D)u\|_0+C'\|\phi u\|_{m-1}\\&\le C\|\psi P(z,D)u\|_0+\frac12\|\psi u\|_m+C'\|\phi u\|_{m-1}\\&\Longrightarrow\frac12\|\psi u\|_m\le C\|\psi P(z,D)u\|_0+C'\|\phi u\|_{m-1}.\end{aligned} \tag{3.7}$$

However, there is a problem here. Namely this is an *a priori* estimate – to move the norm term from right to left we need to know that it is *finite*. Really, that is what we are trying to prove! So more work is required. Nevertheless we will eventually get essentially the same estimate as in the constant coefficient case.

Theorem 3.1. *If $P(z,D)$ is an elliptic differential operator of order m with smooth coefficients in $\Omega\subset\mathbb{R}^n$ and $u\in\mathcal{C}^{-\infty}(\Omega)$ is such that $P(z,D)u\in H^s_{\mathrm{loc}}(\Omega)$ for some $s\in\mathbb{R}$ then $u\in H^{s+m}_{\mathrm{loc}}(\Omega)$ and for any ϕ, $\psi\in\mathcal{C}^\infty_c(\Omega)$ with $\phi=1$ in a neighbourhood of* $\mathrm{supp}(\psi)$ *and $M\in\mathbb{R}$, there exist constants C (depending only on P and ψ) and C' (independent of u) such that*

$$\|\psi u\|_{m+s}\le C\|\psi P(z,D)u\|_s+C'\|\phi u\|_M. \tag{3.8}$$

There are three main things to do. First we need to get the *a priori* estimate first for general s, rather than $s = 0$, and then for general ψ (since up to this point it is only for ψ with sufficiently small support). One problem here is that in the estimates in (3.6) the L^2 norm of a product is estimated by the L^∞ norm of one factor and the L^2 norm of the other. For general Sobolev norms such an estimate does not hold, but something similar does; see Lemma 15.2. The proof of this theorem occupies the rest of this Chapter.

Proposition 3.2. *Under the hypotheses of Theorem 3.1 if in addition $u \in \mathcal{C}^\infty(\Omega)$ then* (3.8) *follows.*

Proof of Proposition 3.2. First we can generalize (3.5), now using Lemma 15.2. Thus, if ψ has support near the point $\bar{z}$

$$\begin{aligned}\|\psi u\|_{s+m} &\le C\|\psi P(\bar{z}, D)u\|_s + \|\phi Q(z, D)\psi u\|_s + C'\|\phi u\|_{s+m-1} \\ &\le C\|\psi P(\bar{z}, D)u\|_s + \delta C\|\psi u\|_{s+m} + C'\|\phi u\|_{s+m-1}.\end{aligned} \tag{3.9}$$

This gives the extension of (3.7) to general s (where we are assuming that u is indeed smooth):

$$\|\psi u\|_{s+m} \le C_s\|\psi P(z, D)u\|_s + C'\|\phi u\|_{s+m-1}. \tag{3.10}$$

Now, given a general element $\psi \in \mathcal{C}^\infty_c(\Omega)$ and $\phi \in \mathcal{C}^\infty_c(\Omega)$ with $\phi = 1$ in a neighbourhood of $\mathrm{supp}(\psi)$ we may choose a partition of unity ψ_j with respect to $\mathrm{supp}(\psi)$ for each element of which (3.10) holds for some $\phi_j \in \mathcal{C}^\infty_c(\Omega)$ where in addition $\phi = 1$ in a neighbourhood of $\mathrm{supp}(\phi_j)$. Then, with various constants

$$\begin{aligned}\|\psi u\|_{s+m} \le \sum_j \|\psi_j u\|_{s+m} &\le C_s \sum_j \|\psi_j \phi P(z, D)u\|_s + C' \sum_j \|\phi_j \phi u\|_{s+m-1} \\ &\le C_s(K)\|\phi P(z, D)u\|_s + C''\|\phi u\|_{s+m-1},\end{aligned} \tag{3.11}$$

where K is the support of ψ and Lemma 15.2 has been used again. This removes the restriction on supports.

Now, to get the full (a priori) estimate (3.8), where the error term on the right has been replaced by one with arbitrarily negative Sobolev order, it is only necessary to iterate (3.11) on a nested sequence of cutoff functions as we did earlier in the constant coefficient case.

This completes the proof of Proposition 3.2. □

So, this proves *a priori* estimates for solutions of the elliptic operator in terms of Sobolev norms. To use these we need to show the regularity of solutions and I will do this by constructing parametrices in a manner very similar to the constant coefficient case.

THEOREM 3.3. *If $P(z,D)$ is an elliptic differential operator of order m with smooth coefficients in $\Omega \subset \mathbb{R}^n$ then there is a continuous linear operator*

$$Q : \mathcal{C}^{-\infty}(\Omega) \longrightarrow \mathcal{C}^{-\infty}(\Omega) \tag{3.12}$$

such that

$$P(z,D)Q = \operatorname{Id} - R_R,\ \ QP(z,D) = \operatorname{Id} - R_L \tag{3.13}$$

where R_R, R_L are properly-supported smoothing operators.

That is, both R_R and R_L have kernels in $\mathcal{C}^\infty(\Omega^2)$ with proper supports. We will in fact conclude that

$$Q : H^s_{\text{loc}}(\Omega) \longrightarrow H^{s+m}_{\text{loc}}(\Omega),\ \forall\ s \in \mathbb{R} \tag{3.14}$$

using the *a priori* estimates.

To construct at least a first approximation to Q essentially the same formula as in the constant coefficient case suffices. Thus consider

$$Q_0 f(z) = \int_\Omega q(z, z - z')\chi(z,z')f(z')dz'. \tag{3.15}$$

Here q is defined as last time, except it now depends on both variables, rather than just the difference, and is defined by inverse Fourier transform

$$q_0(z,Z) = \mathcal{F}^{-1}_{\zeta \longmapsto Z}\hat{q}_0(z,\zeta),\ \ \hat{q}_0 = \frac{1-\chi(z,\zeta)}{P(z,\zeta)} \tag{3.16}$$

where $\chi \in \mathcal{C}^\infty(\Omega \times \mathbb{R})$ is chosen to have compact support in the second variable, so $\operatorname{supp}(\chi) \cap (K \times \mathbb{R}^n)$ is compact for each $K \Subset \Omega$, and to be equal to 1 on such a large set that $P(z,\zeta) \neq 0$ on the support of $1 - \chi(z,\zeta)$. Thus the right side makes sense and the inverse Fourier transform exists.

Next we extend the esimates, (2.21), on the ζ derivatives of such a quotient, using the ellipticity of P. The same argument works for derivatives with respect to z, except no decay occurs. That is, for any compact set $K \Subset \Omega$

$$|D^\beta_z D^\alpha_\zeta \hat{q}_0(z,\zeta)| \le C_{\alpha,\beta}(K)(1+|\zeta|)^{-m-|\alpha|},\ \ z \in K. \tag{3.17}$$

Now the argument, in Lemma 2.5, concerning the singularities of q_0 works with z derivatives as well. It shows that

$$(z_j - z'_j)^{N+k} q_0(z, z-z') \in \mathcal{C}^N(\Omega \times \mathbb{R}^n) \text{ if } k + m > n/2. \tag{3.18}$$

Thus,

$$\operatorname{singsupp} q_0 \subset \operatorname{Diag} = \{(z,z) \in \Omega^2\}. \tag{3.19}$$

The 'pseudolocality' statement (3.19), shows that as in the earlier case, changing the cutoff function in (3.15) changes Q_0 by a properly supported smoothing operator and this will not affect the validity of (3.13) one way or the other! For the moment not worrying too much about how to make sense of (3.15) consider (formally)

$$P(z,D)Q_0f = \int_\Omega \left(P(z,D_Z)q_0(z,Z)\right)_{Z=z-z'}\chi(z,z')f(z')dz' + E_1 f + R_1 f. \tag{3.20}$$

To apply $P(z,D)$ we just need to apply D^α to $Q_0 f$, multiply the result by $p_\alpha(z)$ and add. Applying D_z^α (formally) under the integral sign in (3.15) each derivative may fall on either the 'parameter' z in $q_0(z, z - z')$, the variable $Z = z - z'$ or else on the cutoff $\chi(z, z')$. Now, if χ is ever differentiated the result vanishes near the diagonal and as a consequence of (3.19) this gives a smooth kernel. So any such term is included in R_1 in (3.20) which is a smoothing operator and we only have to consider derivatives falling on the first or second variables of q_0. The first term in (3.20) corresponds to *all* derivatives falling on the second variable. Thus

$$E_1 f = \int_\Omega e_1(z, z - z')\chi(z, z')f(z')dz' \tag{3.21}$$

is the sum of the terms which arise from at least one derivative in the 'parameter variable' z in q_0 (which is to say ultimately the coefficients of $P(z,\zeta)$). We need to examine this in detail. First however notice that we may rewrite (3.20) as

$$P(z,D)Q_0 f = \mathrm{Id} + E_1 + R_1' \tag{3.22}$$

where E_1 is unchanged and R_1' is a new properly supported smoothing operator which comes from the fact that

$$\begin{aligned} P(z,\zeta)q_0(z,\zeta) = 1 - \rho(z,\zeta) \Longrightarrow \\ P(z,D_Z)q_0(z,Z) = \delta(Z) + r(z,Z),\ r \in \mathcal{C}^\infty(\Omega\times\mathbb{R}^n) \end{aligned} \tag{3.23}$$

from the choice of q_0. This part is just as in the constant coefficient case.

So, it is the new error term, E_1 which we must examine more carefully. This arises, as already noted, directly from the fact that the coefficients of $P(z,D)$ are not assumed to be constant, hence $q_0(z,Z)$ depends parameterically on z and this is differentiated in (3.20). So, using Leibniz' formula to get an explicit representation of e_1 in (3.21) we see that

$$e_1(z,Z) = \sum_{|\alpha|\le m,\ |\gamma|<m} p_\alpha(z)\binom{\alpha}{\gamma} D_z^{\alpha-\gamma} D_Z^\gamma q_0(z,Z). \tag{3.24}$$

The precise form of this expansion is not really significant. What *is* important is that at most $m-1$ derivatives are acting on the second variable of $q_0(z,Z)$ since all the terms where all m act here have already been treated. Taking the Fourier transform in the second variable, as before, we find that

$$\hat{e}_1(z,\zeta) = \sum_{|\alpha|\le m,\ |\gamma|<m} p_\alpha(z)\binom{\alpha}{\gamma} D_z^{\alpha-\gamma}\zeta^\gamma \hat{q}_0(z,\zeta) \in \mathcal{C}^\infty(\Omega\times\mathbb{R}^n). \tag{3.25}$$

Thus $\hat{e}_1$ is the sum of products of z derivatives of $q_0(z,\zeta)$ and polynomials in ζ of degree at most $m-1$ with smooth dependence on z. We may therefore transfer the estimates (3.17) to e_1 and conclude that

$$|D_z^\beta D_\zeta^\alpha \hat{e}_1(z,\zeta)| \le C_{\alpha,\beta}(K)(1+|\zeta|)^{-1-|\alpha|}. \tag{3.26}$$

Let us denote by $S^m(\Omega\times\mathbb{R}^n)\subset\mathcal{C}^\infty(\Omega\times\mathbb{R}^n)$ the linear space of functions satisfying (3.17) when $-m$ is replaced by m, i.e.

$$|D_z^\beta D_\zeta^\alpha a(z,\zeta)| \le C_{\alpha,\beta}(K)(1+|\zeta|)^{m-|\alpha|} \Longleftrightarrow a\in S^m(\Omega\times\mathbb{R}^n). \tag{3.27}$$

This allows (3.26) to be written succinctly as $\hat{e}_1\in S^{-1}(\Omega\times\mathbb{R}^n)$.

To summarize so far, we have chosen $\hat{q}_0\in S^{-m}(\Omega\times\mathbb{R}^n)$ such that with Q_0 given by (3.15),

$$P(z,D)Q_0 = \operatorname{Id} + E_1 + R_1' \tag{3.28}$$

where E_1 is given by the same formula (3.15), as (3.21), where now $\hat{e}_1\in S^{-1}(\Omega\times\mathbb{R}^n)$. In fact we can easily generalize this discussion, to do so let me use the notation

$$\operatorname{Op}(a)f(z) = \int_\Omega A(z,z-z')\chi(z,z')f(z')dz', \tag{3.29}$$
$$\text{if } \hat{A}(z,\zeta) = a(z,\zeta)\in S^m(\Omega\times\mathbb{R}^n).$$

PROPOSITION 3.4. *If $a\in S^{m'}(\Omega\times\mathbb{R}^n)$ then*

$$P(z,D)\operatorname{Op}(a) = \operatorname{Op}(pa) + \operatorname{Op}(b) + R \tag{3.30}$$

where R is a (properly supported) smoothing operator and $b\in S^{m'+m-1}(\Omega\times\mathbb{R}^n)$.

PROOF. Follow through the discussion above with $\hat{q}_0$ replaced by a. ☐

So, we wish to get rid of the error term E_1 in (3.21) to as great an extent as possible. To do so we add to Q_0 a second term $Q_1 = \mathrm{Op}(a_1)$ where

$$a_1 = -\frac{1-\chi}{P(z,\zeta)}\hat{e}_1(z,\zeta) \in S^{-m-1}(\Omega \times \mathbb{R}^n). \tag{3.31}$$

Indeed

$$S^{m'}(\Omega \times \mathbb{R}^n)S^{m''}(\Omega \times \mathbb{R}^n) \subset S^{m'+m''}(\Omega \times \mathbb{R}^n) \tag{3.32}$$

(pretty much as though we are multiplying polynomials) as follows from Leibniz' formula and the defining estimates (3.27). With this choice of Q_1 the identity (3.30) becomes

$$P(z,D)Q_1 = -E_1 + \mathrm{Op}(b_2) + R_2,\ b_2 \in S^{-2}(\Omega \times \mathbb{R}^n) \tag{3.33}$$

since $p(z,\zeta)a_1 = -\hat{e}_1 + r'(z,\zeta)$ where $\mathrm{supp}(r')$ is compact in the second variable and so contributes a smoothing operator and by definition $E_1 = \mathrm{Op}(\hat{e}_1)$.

Now we can proceed by induction, let me formalize it a little.

Lemma 3.5. *If $P(z,D)$ is elliptic with smooth coefficients on Ω then we may choose a sequence of elements $a_i \in S^{-m-i}(\Omega\times\mathbb{R}^n)$ $i = 0,1,\dots,$ such that if $Q_i = \mathrm{Op}(a_i)$ then*

$$P(z,D)(Q_0+Q_1+\cdots+Q_j) = \mathrm{Id} + E_{j+1} + R'_j,\ E_{j+1} = \mathrm{Op}(b_{j+1}) \tag{3.34}$$

with R_j a smoothing operator and $b_j \in S^{-j}(\Omega \times \mathbb{R}^n)$, $j = 1,2,\dots.$

Proof. We have already taken the first two steps! Namely with $a_0 = \hat{q}_0$, given by (3.16), (3.28) is just (3.34) for $j = 0$. Then, with a_1 given by (3.31), adding (3.33) to (3.31) gives (3.34) for $j = 1$. Proceeding by induction we may assume that we have obtained (3.34) for some j. Then we simply set

$$a_{j+1} = -\frac{1-\chi(z,\zeta)}{P(z,\zeta)}b_{j+1}(z,\zeta) \in S^{-j-1-m}(\Omega \times \mathbb{R}^n)$$

where we have used (3.32). Setting $Q_{j+1} = \mathrm{Op}(a_{j+1})$ the identity (3.30) becomes

$$P(z,D)Q_{j+1} = -E_{j+1} + E_{j+2} + R''_{j+1},\ E_{j+2} = \mathrm{Op}(b_{j+2}) \tag{3.35}$$

for some $b_{j+2} \in S^{-j-2}(\Omega \times \mathbb{R}^n)$. Adding (3.35) to (3.34) gives the next step in the inductive argument. □

Consider the error term in (3.34) for large j. From the estimates on an element $a \in S^{-j}(\Omega \times \mathbb{R}^n)$

$$|D_z^\beta D_\zeta^\alpha a(z,\zeta)| \le C_{\alpha,\beta}(K)(1+|\zeta|)^{-j-|\alpha|} \tag{3.36}$$

it follows that if $j > n + k$ then $\zeta^\gamma a$ is integrable in ζ with all its z derivatives for $|\zeta| \le k$. Thus the inverse Fourier transform has continuous derivatives in all variables up to order k. Applied to the error term in (3.34) we conclude that

$$E_j = \mathrm{Op}(b_j) \text{ has kernel in } \mathcal{C}^{j-n-1}(\Omega^2) \text{ for large } j. \tag{3.37}$$

Thus as j increases the error terms in (3.34) have increasingly smooth kernels.

Now, standard properties of operators and kernels, see Lemma 17.1, show that operator

$$Q_{(k)} = \sum_{j=0}^{k} Q_j \tag{3.38}$$

comes increasingly close to satisfying the first identity in (3.13), except that the error term is only finitely (but arbitrarily) smoothing. Since this is enough for what we want here I will banish the actual solution of (3.13) to the addenda to this Chapter.

LEMMA 3.6. *For k sufficiently large, the left parametrix $Q_{(k)}$ is a continuous operator on $\mathcal{C}^\infty(\Omega)$ and*

$$Q_{(k)} : H^s_{\mathrm{loc}}(\Omega) \longrightarrow H^{s+m}_{\mathrm{loc}}(\Omega) \ \forall\ s \in \mathbb{R}. \tag{3.39}$$

PROOF. So far I have been rather cavalier in treating $\mathrm{Op}(a)$ for $a \in S^m(\Omega \times \mathbb{R}^n)$ as an operator without showing that this is really the case, however this is a rather easy exercise in distribution theory. Namely, from the basic properties of the Fourier transform and Sobolev spaces

$$A(z, z - z') \in \mathcal{C}^k(\Omega; H^{-n-1+m-k}_{\mathrm{loc}}(\Omega)) \ \forall\ k \in \mathbb{N}. \tag{3.40}$$

It follows that $\mathrm{Op}(a) : H^{n+1-m+k}_{\mathrm{c}}(\Omega)$ into $\mathcal{C}^k(\Omega)$ and in fact into $\mathcal{C}^k_{\mathrm{c}}(\Omega)$ by the properness of the support. In particular it does define an operator on $\mathcal{C}^\infty(\Omega)$ as we have been pretending and the steps above are easily justified.

A similar argument, which I will not give here since it is better to do it by duality (see the addenda), shows that for any fixed s

$$A : H^s_{\mathrm{loc}}(\Omega) \longrightarrow H^S_{\mathrm{loc}}(\Omega) \tag{3.41}$$

for some S. Of course we want something a bit more precise than this.

If $f \in H^s_{\mathrm{loc}}(\Omega)$ then it may be approximated by a sequence $f_j \in \mathcal{C}^\infty(\Omega)$ in the topology of $H^s_{\mathrm{loc}}(\Omega)$, so $\mu f_j \to \mu f$ in $H^s(\mathbb{R}^n)$ for each $\mu \in \mathcal{C}^\infty_c(\Omega)$. Set $u_j = Q_{(k)} f_j \in \mathcal{C}^\infty(\Omega)$ as we have just seen, where k is fixed but will be chosen to be large. Then from our identity $P(z,D)Q_{(k)} = \operatorname{Id} + R_{(k)}$ it follows that

$$P(z,D)u_j = f_j + g_j,\ g_j = R_{(k)}f_j \to R_{(k)}f \text{ in } H^N_{\text{loc}}(\Omega) \tag{3.42}$$

for k large enough depending on s and N. Thus, for k large, the right side converges in $H^s_{\text{loc}}(\Omega)$ and by (3.41), $u_j \to u$ in some $H^S_{\text{loc}}(\Omega)$. But now we can use the *a priori* estimates (3.8) on $u_j \in \mathcal{C}^\infty(\Omega)$ to conclude that

$$\|\psi u_j\|_{s+m} \le C\|\psi(f_j + g_j)\|_s + C''\|\phi u_j\|_S \tag{3.43}$$

to see that ψu_j is bounded in $H^{s+m}(\mathbb{R}^n)$ for any $\psi \in \mathcal{C}^\infty_c(\Omega)$. In fact, applied to the difference $u_j - u_l$ it shows the sequence to be Cauchy. Hence in fact $u \in H^{s+m}_{\text{loc}}(\Omega)$ and the estimates (3.8) hold for this u. That is, $Q_{(k)}$ has the mapping property (3.39) for large k. □

In fact the continuity property (3.39) holds for all $\operatorname{Op}(a)$ where $a \in S^m(\Omega \times \mathbb{R}^n)$, not just those which are parametrices for elliptic differential operators. I will comment on this below – it is one of the basic results on pseudodifferential operators.

There is also the question of the second identity in (3.13), at least in the same finite-order-error sense. To solve this we may use the transpose identity. Thus taking formal transposes this second identity should be equivalent to

$$P^tQ^t = \operatorname{Id} - R^t_L. \tag{3.44}$$

The transpose of $P(z,D)$ is the differential operotor

$$P^t(z,D) = \sum_{|\alpha|\le m} (-D)^\alpha_z p_\alpha(z). \tag{3.45}$$

This is again of order m and after a lot of differenttiation to move the coefficients back to the left we see that its leading part is just $P_m(z,-D)$ where $P_m(z,D)$ is the leading part of $P(z,D)$, so it is elliptic in Ω exactly when P is elliptic. To construct a solution to (3.45), up to finite order errors, we need just apply Lemma 3.5 to the transpose differential operator. This gives $Q'_{(N)} = \operatorname{Op}(a'_{(N)}$ with the property

$$P^t(z,D)Q'_{(N)} = \operatorname{Id} - R'_{(N)} \tag{3.46}$$

where the kernel of $R'_{(N)}$ is in $C^N(\Omega^2)$. Since this property is preserved under transpose we have indeed solved the second identity in (3.13) up to an arbitrarily smooth error.

Of course the claim in Theorem 3.3 is that the one operator satisfies both identities, whereas we have constructed two operators which each satisfy one of them, up to finite smoothing error terms

$$P(z,D)Q_R = \operatorname{Id} - R_R,\ Q_L P(z,D) = \operatorname{Id} - R_L. \tag{3.47}$$

However these operators must themselves be equal up to finite smoothing error terms since composing the first identity on the left with Q_L and the second on the right with Q_R shows that

$$Q_L - Q_L R_R = Q_L P(z,D) Q_R = Q_R - R_L Q_R \tag{3.48}$$

where the associativity of operator composition has been used. We have already checked the mapping property(3.39) for both Q_L and Q_R, assuming the error terms are sufficiently smoothing. It follows that the composite error terms here map $H^{-p}_{\mathrm{loc}}(\Omega)$ into $H^{p}_{\mathrm{loc}}(\Omega)$ where $p \to \infty$ with k with the same also true of the transposes of these operators. Such an operator has kernel in $C^{p'}(\Omega^2)$ where again $p' \to \infty$ with k. Thus the difference of Q_L and Q_R itself becomes arbitrarily smoothing as $k \to \infty$.

Finally then we have proved most of Theorem 3.3 except with arbitrarily finitely smoothing errors. In fact we have not quite proved the regularity statement that $P(z,D)u \in H^s_{\mathrm{loc}}(\Omega)$ implies $u \in H^{s+m}_{\mathrm{loc}}(\Omega)$ although we came very close in the proof of Lemma 3.6. Now that we know that $Q_{(k)}$ is also a right parametrix, i.e. satisfies the second identity in (3.8) up to arbitrarily smoothing errors, this too follows. Namely from the discussion above $Q_{(k)}$ is an operator on $\mathcal{C}^{-\infty}(\Omega)$ and

$$Q_{(k)} P(z,D) u = u + v_k,\ \psi v_k \in H^{s+m}(\Omega)$$

for large enough k so (3.39) implies $u \in H^{s+m}_{\mathrm{loc}}(\Omega)$ and the *a priori* estimates magically become real estimates on all solutions.

Addenda to Chapter 4

Asymptotic completeness to show that we really can get smoothing errors.

Some discussion of pseudodifferential operators – adjoints, composition and boundedness, but only to make clear what is going on.

Some more reassurance as regards operators, kernels and mapping properties – since I have treated these fairly shabbily!