Bidiagonal Decompositions of Oscillating Systems of Vectors

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Abstract

We establish necessary and sufficient conditions, in the language of bidiagonal decompositions, for a matrix \( V \) to be an eigenvector matrix of a totally positive matrix. Namely, this is the case if and only if \( V \) and \( V^{-T} \) are lowerly totally positive.

These conditions translate into easy positivity requirements of the parameters in the bidiagonal decomposition of \( V \). Using these decompositions we give elementary proofs of the oscillating properties of \( V \). In particular, the fact that the \( j \)th column of \( V \) has \( j - 1 \) changes of sign.

Our new results include the fact that the \( Q \) matrix in a QR decomposition of a totally positive matrix belongs to the above class (and thus has the same oscillating properties).

Numerical issues are also discussed.

1 Introduction

The matrices with all minors positive are called totally positive (TP). In this paper we consider the class of matrices that are eigenvector matrices of TP matrices. We denote this class as ETP. The TP matrices are diagonalizable and their eigenvalues are positive and distinct. Therefore we further assume that the columns of each ETP matrix are permuted so that the \( j \)th column is an eigenvector corresponding to the \( j \)th largest eigenvalue.

The utilization of bidiagonal decompositions as means of studying the properties of TP and related matrices has been particularly prominent recently [6, 7, 10, 13, 14]. Here we take the same approach with the ETP matrices. We establish a classification of the ETP matrices in the language of their bidiagonal decompositions. In particular, we prove that a matrix \( V \) is ETP if the multipliers needed to eliminate the lower triangular parts of \( V \) and \( V^{-T} \) in the process of Neville elimination are positive (see Sections 2 and 3 for the formal definitions of these notions).

Using bidiagonal decompositions we give new elementary proofs of the oscillating properties of ETP matrices. In particular, the fact that the \( j \)th column of any ETP matrix has exactly \( j - 1 \) changes of sign.

The above classification allows us to obtain full parameterization of the orthogonal ETP matrices. In particular, any matrix \( Q \) from the QR decomposition of a TP matrix is ETP and is parameterized by the \( n(n - 1)/2 \) nontrivial parameters in the lower bidiagonal factors of the bidiagonal decomposition of \( Q \). As an eigenvector matrix of some (symmetric) TP matrix, its \( j \)th column has exactly \( j - 1 \) changes of sign.

This parameterization of \( Q \) is also important numerically; the \( n(n - 1)/2 \) nontrivial parameters determine the directions of the columns of \( Q \) very accurately and allow for their accurate computation in the appropriate sense (see Section 7).

The paper is organized as follows. In Section 2 we survey the bidiagonal decompositions of TP matrices. In Section 3 we survey matrices whose lower triangular factors are (lowerly) totally positive. We establish that a matrix \( V \) is ETP if \( V \) and \( V^{-T} \) are lowerly totally positive in Section 4. We give elementary proofs

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of the oscillating properties of TP and ETP matrices in Sections 5 and 6. Finally, in Section 7, we consider numerical issues and stability.

Note: We use MATLAB [18] notation for submatrices.

2 Background

In this section we review the properties of the bidiagonal decompositions of matrices and refer the reader to [6, 7, 13, 14] for more details on the main ideas, which can be traced back to Whitney [20].

The bidiagonal decomposition of a matrix \( A \) is obtained by eliminating it (in a process called Neville elimination) using adjacent rows and columns. Each (row) elimination step is equivalent to a multiplication on the left by a matrix of the form

\[
E_i(x) = \begin{bmatrix}
1 & & & \\
& \ddots & \ & \\
& & x & \ \\
& & \ & 1
\end{bmatrix},
\]

which differs from the identity only in the \((i, i-1)\) entry.

The lower triangular part of a matrix (call it \( A \)) is eliminated one subdiagonal at a time, starting with the \((n, 1)\) entry. Once the matrix is reduced to an upper triangular form, the same process is applied by columns resulting in the decomposition:

\[
A = \prod_{i=1}^{n-1} \prod_{j=n-i+1}^{n} E_j(b_{j,i+j-n}) \cdot D \cdot \left( \prod_{i=n-1}^{1} \prod_{j=n-i+1}^{n} E_j^T(b_{i+j-n,j}) \right),
\]

where \( D = \text{diag}(b_{11}, b_{22}, \ldots, b_{nn}) \). In the notation of (1) and throughout this paper, \( \prod_{i=1}^{n-1} \) indicates that the product is taken for \( i \) from \( n-1 \) down to 1. Although somewhat nonstandard, this notation allows us to preserve the symmetry in (1).

According to a result of Gasca and Peña [13, Theorem 4.3], \( A \) is TP if and only if (1) exists, it is unique, and \( b_{ij} > 0 \), \( i, j = 1, 2, \ldots, n \).

If we group the factors in the parentheses of (1) into factors \( L \) and \( U \) then (1) becomes \( A = LDU \)—the LDU decomposition of \( A \) from Gaussian elimination (with no pivoting).

3 LTP matrices

In this section we consider matrices for which the positivity constraints are only imposed on the lower bidiagonal factors in (1).

Definition 1 (LTP matrix) A matrix is called lowerly totally positive (LTP) if it is nonsingular, its LDU decomposition exists, and all nontrivial minors of the \( L \) factor in that decomposition are positive (i.e., all minors \( \det \left( L([i_1, i_2, \ldots, i_p], [j_1, j_2, \ldots, j_p]) \right) \) such that \( i_1 \geq j_1, i_2 \geq j_2, \ldots, i_p \geq j_p \)).

In other words, \( A = LDU \) is LTP if and only if all multipliers \( b_{ij} \) for \( i > j \) in the decomposition (1) of \( L \) are positive. The \( D \) factor can have any (nonzero) entries on the diagonal.

This definition differs from the one used in Cryer [5] and Gasca and Peña [11] in that we do not impose any positivity restrictions on the \( D \) factor in this decomposition (other than to be nonsingular). This way right diagonal scaling does not affect the LTP structure of a matrix in very much the same way in which right diagonal scaling of an ETP matrix does not affect the ETP structure of that matrix.

In our proofs it will be convenient to sometimes reverse the rows and columns of a matrix or to take an inverse and make all its entries positive. These operations preserve the TP structure and we define them formally here.
\textbf{Definition 2 (Taking a converse and re-signing)} If \( A = [a_{ij}]_{i,j=1}^n \) then the matrices 
\[
A^\# \equiv [a_{n-i+1,n-j+1}]_{i,j=1}^n \quad \text{and} \quad A^* \equiv [(-1)^{i+j}a_{ij}]_{i,j=1}^n
\]
are called the converse of \( A \) and the re-signed inverse of \( A \), respectively.

It is easily verified that the four operations: re-signing, taking a converse, inverse, or a transpose, commute.

A matrix \( A \) is TP if and only if \( A^\# \) is TP, and also if and only if \((A^{-1})^*\) is TP \cite[Proposition 5\textsuperscript{o}, p. 75]{8}.

The factors \( L \) and \( U \) in the LDU decomposition of a TP matrix inherit the total positivity properties with respect to their nontrivial minors, which properties are also preserved under taking a converse or a re-signed inverse as we now prove.

\textbf{Lemma 1} A unit lower triangular matrix \( L \) is LTP if and only if \((L^\#)^T\) is LTP.

\textbf{Proof:} Clearly \( L \) and \((L^\#)^T\) are simultaneously unit lower triangular. The nontrivial minors of \( L \) and those of \((L^\#)^T\) are positive simultaneously since
\[
L([i_1,i_2,\ldots,i_p],[j_1,j_2,\ldots,j_p]) = (L^\#)^T([n+1-j_p,\ldots,n+1-j_1],[n+1-i_p,\ldots,n+1-i_1]),
\]
where \( 1 \leq i_1 < i_2 < \cdots < i_p \leq n, 1 \leq j_1 < j_2 < \cdots < j_p \), and \( i_1 \geq j_1, i_2 \geq j_2, \ldots, i_p \geq j_p \). \( \Box \)

\textbf{Lemma 2} A unit lower triangular matrix \( L \) is LTP if and only if its re-signed inverse \((L^{-1})^*\) is LTP.

\textbf{Proof:} This result follows from applying the argument in Gantmacher and Krein \cite[Proposition 5\textsuperscript{o}, p. 75]{8} to the nontrivial minors of \( L \) and those of \((L^{-1})^*\). \( \Box \)

\textbf{Corollary 1} If \( A \) is a nonsingular matrix, then \( A^{-T} \) is LTP if and only if \( A^\# \) is LTP.

\begin{proof} \end{proof}

\textbf{Lemma 3} Let \( A = UDL \) be the UDL decomposition of an \( n \times n \) matrix \( A \). Then \( A \) is TP if and only if \( L \) and \( U^T \) are LTP and \( D \) has a positive diagonal \cite{10}. An analogous result for the UDL decomposition also holds.

\begin{proof} \end{proof}

\textbf{Theorem 1} Let the TP matrix \( A \) have eigenvalues \( \lambda_1 > \lambda_2 > \cdots > \lambda_n > 0 \). Then \( A \) is similar to the bidiagonal matrix
\[
B = \begin{bmatrix}
\lambda_1 & \lambda_1 - \lambda_2 & \lambda_1 - \lambda_3 & \cdots & \lambda_1 - \lambda_n \\
\lambda_2 & \lambda_2 - \lambda_3 & \cdots & \lambda_2 - \lambda_n \\
\lambda_3 & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots \\
\lambda_{n-1} & \lambda_{n-2} & \cdots & \lambda_{n-1} - \lambda_n & \lambda_n
\end{bmatrix}
\]
via an LTP similarity transformation.
4 \ LTP^2 matrices

In this section we prove that a matrix $V$ is ETP if and only if $V$ and $V^{-T}$ are LTP.

**Definition 3** We call a matrix $V$ doubly lowerly totally positive (denoted as LTP\(^2\)) if $V$ and $V^{-T}$ are LTP.

From Corollary 1 we have that, equivalently, $V$ is LTP\(^2\) if $V$ and $V^{*\#}$ are LTP.

The following lemma establishes that for an LTP\(^2\) matrix, the $D$ factors in the LDU decompositions of $A$ and $A^{-T}$ have the same sign patterns. This property will be critical in the proof of Theorem 2 below.

**Lemma 4** Let $A$ be LTP\(^2\) and let $A = LDU$ and $A^{-T} = \bar{L}ar{D}\bar{U}$ be the LDU decompositions of $A$ and $A^{-T}$, respectively. Then $D_{ii}\bar{D}_{ii} > 0$ for $i = 1, 2, \ldots, n$.

**Proof:** We have $D^{-1}U^{-T} = L^T A^{-T} = (L^T \bar{L}).(\bar{D}\bar{U}) = L'D'U'D\bar{U}$, where $L'D'U'$ is the LDU decomposition of the TP matrix $L^T \bar{L}$. Therefore

$$\left(D^{-1}U^{-T} D\right) \cdot D^{-1} \cdot I = L' \cdot (D'\bar{D}) \cdot (\bar{D}^{-1}U'\bar{D}\bar{U}).$$

Since both sides of (2) are LDU decompositions of the same matrix, the corresponding factors must be equal. In particular, $D^{-1} = D'\bar{D}$ and the result follows. \(\square\)

**Theorem 2** A matrix $V$ is LTP\(^2\) if and only if it is ETP.

**Proof:** Let $A$ be an $n \times n$ TP matrix. According to Theorem 1, $A$ is similar to a bidiagonal matrix $B$ via an LTP similarity transformation matrix $S$. The eigenvector matrix of $B$ (call it $Z$) is upper triangular, thus $V \equiv SZ$ is LTP ($V$ and $S$ share the $L$ factor in their LDU decompositions). Given the order of the distinct eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$ of $A$, the only source of nonuniqueness of the eigenvector matrix comes from right diagonal scaling, which does not affect the $L$ factor. Thus every ETP matrix is LTP.

On the other side, since $V^{-T}$ is an eigenvector matrix of the TP matrix $A^T$, it is also LTP.

Conversely, if $V$ is an LTP\(^2\) matrix, we follow the proof of Theorem 19 on p. 272 in Gantmacher and Krein [8].

Let $A = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where the $\lambda_i$ are such that $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$ but otherwise arbitrary.

We will prove that some (high enough) power of $A = VAV^{-1}$ is TP.

Let $F = A^m = V\Lambda^m V^{-1}$. From the Cauchy–Binet identity we have for any minor of $F$:

$$\det \left(F\left([i_1, i_2, \ldots, i_p], [k_1, k_2, \ldots, k_p]\right)\right)$$

$$= \sum_{1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_p \leq n} \lambda^{m}_{\alpha_1} \lambda^m_{\alpha_2} \cdots \lambda^m_{\alpha_p}$$

$$\times \det \left(V\left([i_1, i_2, \ldots, i_p], [\alpha_1, \alpha_2, \ldots, \alpha_p]\right)\right) \cdot \det \left(V^{-1}\left([\alpha_1, \alpha_2, \ldots, \alpha_p], [k_1, k_2, \ldots, k_p]\right)\right),$$

where $1 \leq i_1 < i_2 < \cdots < i_p \leq n$ and $1 \leq k_1 < k_2 < \cdots < k_p \leq n$. For large enough $m$, the sign of this minor will be dominated by its leading term

$$\lambda^m_1 \lambda^m_2 \cdots \lambda^m_p \cdot \det \left(V\left([i_1, i_2, \ldots, i_p], [1, 2, \ldots, p]\right)\right) \cdot \det \left(V^{-T}\left([k_1, k_2, \ldots, k_p], [1, 2, \ldots, p]\right)\right),$$

which is positive (since $V$ and $V^{-T}$ are LTP and the $D$ factors in the LDU decompositions of $V$ and $V^{-T}$ have the same sign pattern on the diagonal—see Lemma 4). \(\square\)

The following theorem describes the properties of the upper triangular factor of an LTP\(^2\) matrix.

**Theorem 3** If $A = LDU$ is LTP\(^2\) and $D$ has a positive diagonal, then $U^{-T}$ is LTP.

**Proof:** With the notation as in Lemma 4, $U^{-T} = L'D'U'D\bar{U}$. By comparing the lower triangular factors we get $U^{-T} = L'$ is LTP. \(\square\)

Note: A matrix $A = LDU$ such that $L$ and $U^{-T}$ are LTP and $D$ has a positive diagonal is called a $\gamma$-matrix [11]. The $\gamma$-matrix property is necessary, but not sufficient to characterize the eigenvector matrices or the $Q$ factors of TP matrices.

\(^1\)Note that we switched from $V^{-1}$ to $V^{-T}$, also reversing the order of the indices, in going from (3) to (4).
Corollary 2 An orthogonal matrix $Q$ is LTP if and only if it is a $Q$ factor in a QR decomposition of a square TP matrix.

Proof: If $A = QR$ and $R$ is upper triangular, then $A$ and $Q$ share the $L$ factor in their respective LDU decompositions. Therefore $Q$ is LTP. Conversely, if $Q = LDU$ is orthogonal and LTP, then $Q$ is the $Q$ factor in the QR decomposition of any TP matrix $LDU$, where $D$ is diagonal with positive entries on the diagonal and $U$ is unit upper triangular and upper totally positive (i.e., $U^{-T}$ is LTP).

Since $Q$ is orthogonal, $Q^{-T} = Q$. Thus $Q$ is LTP$^2$. □

In particular, $Q$ has the same oscillating properties as the eigenvector matrices of TP matrices and the number of sign changes in the $j$th column of $Q$ is exactly $j - 1$ (see Section 6).

Theorem 2 implies the following about the structure of orthogonal LTP matrices.

Corollary 3 Every LTP orthogonal matrix $Q$ is the eigenvector matrix of some symmetric TP matrix. Therefore the eigenvector matrices of the symmetric TP matrices are thus parameterized by the $n(n - 1)/2$ positive multipliers $b_{ij} > 0, i > j$ in the bidiagonal decomposition of $Q$.

5 Variation-diminishing properties of TP matrices

The bidiagonal decompositions of nonsingular TN matrices can be used to obtain easy proofs of certain well known (see, e.g., Gantmacher and Krein [8]) variation-diminishing properties for these matrices. In particular, a multiplication by a nonsingular TN matrix does not increase the number of sign changes in a vector. This result is mentioned without proof in [6, Section 4.4], and we are unaware of a proof (using bidiagonal decompositions) anywhere else. For completeness, we include one here.

Next, we recall the definition of number of sign changes in a vector.

Definition 4 (Number of sign changes) [8, p. 86] Let $u_1, u_2, \ldots, u_n$ be a sequence of numbers. If some of the numbers are zero, then we can assign them arbitrary signs. The number of sign changes of the above sequence is then the number of instances for $i = 1, 2, \ldots, n - 1$ where $u_i$ and $u_{i+1}$ have different signs.

Depending on our choice of signs of the zero components we can have a different number of sign changes. We define $S_u^-$ and $S_u^+$ to be the minimum and maximum number of sign changes among all possible choices of signs of the zero components.

If $S_u^- = S_u^+$, we say that the (exact) number of sign changes of the vector $u$ is $S_u = S_u^- = S_u^+$.

Next, we establish that the simplest nontrivial TN matrix, $E_1(x), x > 0$, does not diminish the maximum number of sign changes in a vector.

Lemma 5 Let $u$ be an $n$-vector and $w = E_1(x) \cdot u$, where $x > 0$. Then $S_u^+ \leq S_w^+.$

Proof: We have $w_j = u_j$ for $j = 1, 2, \ldots, i - 1, i + 1, \ldots, n$, and $w_i = u_i + xu_{i-1}$.

We assume that $u_{i-1} \neq 0$, otherwise $w = u$ and the claim is trivially true. Without loss of generality we can also assume that $u_{i-1} > 0$, since $S_u^+ = S_w^+$.

Consider first the case $u_i \geq 0$. Then we have $w_i > 0$. We assign the same signs to the entries of $u$ as the corresponding signs assigned to the entries of $w$ in the computation of $S_w^+$. (We can always do this. We have $u_j = w_j, j \neq i$. The $i$th entry of $w, w_i > 0$, is counted with a positive sign in $S_w^+.$ We thus assign the same (positive) sign to $u_i \geq 0$.) With signs thus assigned $u$ has $S_u^+$ sign changes. Therefore $S_w^+ \leq S_u^+$.

Alternatively, we consider the case $u_i < 0$. We can assume that $u_i$ is counted in $S_u^+$ with a positive sign (otherwise $S_u^+ = S_w^+$). For $j = 1, 2, \ldots, i - 1, i + 1, \ldots, n$, we assign to $u_i$ the same sign as the one assigned to $w_i$ in the computation of $S_w^+$. Denote by $S$ the thus obtained number of sign changes in $u$. We claim that $S_w^+ \leq S \leq S_u^+$. The second inequality is obvious. To see that the first one is true, we consider the sequences

$$\text{sign}(u_{i-1}, u_i, u_{i+1}) = (+, -, *) \quad \text{and} \quad \text{sign}(w_{i-1}, w_i, w_{i+1}) = (+, +, *),$$

where the “$*$” stands for the same sign (“+” or “−”) assigned to both $u_{i+1} = w_{i+1}$. If $\text{sign}(u_{i+1}) = \text{sign}(w_{i+1}) = +$, then $S_w^+ + 2 = S$. If $\text{sign}(u_{i+1}) = \text{sign}(w_{i+1}) = −$, then $S_w^+ = S$.

We are done. □

By applying the above lemma repeatedly, we obtain the following corollary.
Corollary 4 (TN Variation Diminishing Property) For any real n-vector \( u \) and a nonsingular TN matrix \( A \), \( S_{Au}^+ \leq S_u^+ \) and \( S_{Au}^- \leq S_u^- \).

**Proof:** According to Theorem 4.2 in [13], \( A \) can be uniquely factored in the form (1) with \( b_{ii} > 0, i = 1, 2, \ldots, n \) and \( b_{ij} \geq 0, i \neq j \). Then first assertion follows by repeatedly applying Lemma 5.

To prove the second, let \( J = \text{diag}(-1, 1, -1, \ldots, (-1)^n) \). If \( w = Au \) then \( Ju = (A^{-1})^*w \), where \( (A^{-1})^* = (JAJ)^{-1} \) is the re-signed inverse of \( A \), which is also nonsingular and TN (see, e.g., [8, Proposition 5]).

From Lemma 5, \( S_{Jw}^+ = S_{(A^{-1})^*}^- \leq S_{Jw}^- \). Since for any n-vector \( v \), \( S_v^+ + S_Jv = n - 1 \), we have \( S_u^- \geq S_{w}^- = S_{Au}^- \). □

When \( A \) is TP a much stronger result is true: \( S_{Au}^\leq S_u^- \). To prove it, we need the following lemma.

**Lemma 6** If \( A \) is TP, then for every \( i, 2 \leq i \leq n \), there exist TP matrices \( B \) and \( C \), and positive numbers \( x \) and \( y \) such that

\[
A = B \cdot E_i(x) \quad \text{and} \quad A = C \cdot E_i(y).
\]

**Proof:** This lemma is nearly obvious if we use a limiting argument: From Cauchy–Binet, any minor of \( B = A \cdot E_i(x) \) is a linear function of \( x \), say \( ax + b \), where \( b > 0 \) is the value of the same minor of \( A \). Clearly, for small enough \( x \) all minors of \( B \) will be positive. The second claim follows by applying the same argument to \( A^\# \). □

However, in the spirit of the rest of this paper, in the Appendix we give a second, constructive proof based on bidiagonal decompositions, by providing an explicit way to extract an \( E_i(x) \) factor out of \( A \).

Theorem 4 (TP Variation Diminishing Property) If \( A \) is TP and \( u \) is a nonzero n-vector, then \( S_{Au}^+ \leq S_u^- \).

**Proof:** One way to compute \( S_u^- \) is to count any zero (say \( u_i \)) in \( u \) with the same sign as the sign of the first nonzero component in \( u \) following \( u_i \). The trailing zero components of \( u \) can be counted with the same sign as the last nonzero in \( u \).

The main idea here is to factor appropriate \( E_i(x) \) and \( E_i^T(y) \) out of \( A \) and factor them into \( u \) to make the zero components of \( u \) nonzeros with the same signs as in the above computation of \( S_u^- \).

The construction is straightforward.

If \( u_{i-1} = 0 \) but \( u_i \neq 0 \), we use Lemma 6 to write \( A = F \cdot E_i^T(y) \) for some \( y > 0 \), where \( F \) is TP. If \( v = E_i(y)u \), then \( v_{i-1} = yu_i \) (the magnitude of \( y \) is unimportant, its only purpose is to make \( v_{i-1} \) have the same sign as \( u_i \)). We continue this process until we obtain a vector (call it \( p \)) which can only have zero components at the end, say \( p_k = \cdots = p_n = 0 \), but \( p_{k-1} \neq 0 \). By factoring out \( E_j(x_j), j = k, k+1, \ldots, n \) and factoring them into \( p \) we obtain a new vector (call it \( q \)) which has no zero components and \( S_q = S_q^+ = S_q^- \). If \( A = BC \) where \( q = Cu \), then \( w = Au = BCu = Bq \). By applying Corollary 4 we obtain \( S_u^+ \leq S_q^- \). □

6 Oscillating properties of ETP matrices

In this section we use the bidiagonal decompositions of ETP matrices to establish their oscillating properties. We note that oscillatory are matrices some of whose power is TP [8]. For this reason we have chosen the ETP notation for these matrices and call them oscillating systems of vectors in the title.

**Theorem 5** Let the \( n \times n \) matrix \( A \) be LTP and let

\[
u = c_1a_1 + c_2a_2 + \cdots + c_ka_k,
\]

where \( c_1, c_2, \ldots, c_k \) are arbitrary real constants such that \( c_k \neq 0 \), and \( a_1, a_2, \ldots, a_n \) are the columns of \( A \). Then

\[
S_u^+ \leq k - 1.
\]

\(^2\)Additionally, \( b_{ij}, i \neq j \) must satisfy \( b_{ij} = 0 \) if \( b_{i-1,j} = 0 \) and \( i > j \), and \( b_{ij} = 0 \) if \( b_{i,j-1} = 0 \) and \( i < j \). However, these conditions are unimportant here.
Proof: We have \( u = Ac \), where \( c = (c_1, c_2, \ldots, c_k, 0, \ldots, 0)^T \).

The matrices \( E_i(x) \) satisfy \( E_i(x)E_j(y) = E_j(y)E_i(x) \), unless \(|i - j| = 1\). Thus we can re-order some of the factors in (1) to obtain

\[
A = \left( \prod_{i=1}^{n-1} \prod_{j=i+1}^n E_j(b_{ij}) \right) \cdot D \cdot \left( \prod_{i=1}^{n-1} \prod_{j=i+1}^n E_j^T(b_{ij}) \right) \equiv LDU.
\]

We can think of (5) as having been obtained by performing Neville elimination one column at a time followed by analogous elimination of the rows. Clearly the same multipliers would be used.

Let \( v = DUC \). Since \( c_k \neq 0 \) and \( c_i = 0 \) for \( i > k \), we have \( v = (v_1, v_2, \ldots, v_k, 0, 0, \ldots, 0) \), \( v_k \neq 0 \), i.e., \( v_i = 0 \) for \( i > k \). In turn the condition \( v_i = 0 \) for \( i > k \) implies \( E_i(x) \cdot v = v \) for \( i > k + 1 \), and thus

\[
u = Ac = Lv = \left( \prod_{i=1}^{n-1} \prod_{j=i+1}^n E_j(b_{ji}) \right) \cdot v = \left( \prod_{i=1}^{k-1} \prod_{j=i+1}^n E_j(b_{ji}) \right) \cdot w.
\]

We have \( w_i = v_i \) for \( i = 1, 2, \ldots, k \) and \( w_i = b_{ik}b_{i-1,k} \cdots b_{k+1,k}v_k \neq 0 \) for \( i > k \). Since \( b_{ij} > 0 \) for \( i > j \), the entries \( w_k, w_{k+1}, \ldots, w_n \) have the same sign (and are nonzero). Therefore \( \sum_{i=1}^n w_i \leq k - 1 \).

Now

\[
u = \left( \prod_{i=1}^{k-1} \prod_{j=i+1}^n E_j(b_{ji}) \right) \cdot w;
\]

and Lemma 5 implies \( \sum_{i=1}^n w_i \leq \sum_{i=1}^n w_i = k - 1 \). □

**Theorem 6** Let the \( n \times n \) matrix \( A \) be LTP, and let \( a_j, j = 1, 2, \ldots, n \) be the columns of \( A \). If \( c_i, c_{i+1}, \ldots, c_j, \left( \sum_{k=i}^j c_k^2 \neq 0 \right) \) is a sequence of real numbers, then the number of sign changes of

\[
u = c_i a_i + c_i a_{i+1} + \ldots + c_j a_j
\]

lies between \( i - 1 \) and \( j - 1 \), i.e.,

\[i - 1 \leq S_u^- \leq S_u^+ \leq j - 1.
\]

In particular, the number of sign changes in \( a_j \) is exactly \( j - 1 \). □

**Proposition 1** The entries \( b_{ij} \) of the bidiagonal decomposition of an orthogonal LTP matrix \( Q \) are accurately determined by the “lower triangular part,” i.e., by the entries \( b_{ij}, i > j \).

Proof: Let \( Q = LDU \) be the LDU decomposition of \( Q \). Then \( Q^{-T} = Q \) implies \( LDU = L^{-T}D^{-T}U^{-T} \) or equivalently,

\[
L^T L = D^{-T}U^{-T}U^{-1}D^{-1}.
\]

If the LDU decomposition of the TP matrix \( L^T L \) is \( L^T L = \bar{L}\bar{D}\bar{U} \), then

\[
\bar{L}\bar{D}\bar{U} = D^{-T}U^{-T}U^{-1}D^{-1},
\]

7 Stability

**Proposition 1** The entries \( b_{ij} \) of the bidiagonal decomposition of an orthogonal LTP matrix \( Q \) are accurately determined by the “lower triangular part,” i.e., by the entries \( b_{ij}, i > j \).

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If the LDU decomposition of the TP matrix \( L^T L \) is \( L^T L = \bar{L}\bar{D}\bar{U} \), then

\[
\bar{L}\bar{D}\bar{U} = D^{-T}U^{-T}U^{-1}D^{-1},
\]
meaning \( U^{-1} D^{-1} = \tilde{D}^{1/2} U' \). By comparing the diagonal, \( D^{-1} = \tilde{D}^{1/2} \). Thus \( U = \tilde{D}^{1/2} U^{-1} \tilde{D}^{-1/2} \).

From Cauchy–Binet each entry in the bidiagonal decompositions of \( L, \tilde{D}, U \) is a linear function with positive coefficients in \( b_{ij}, i > j \) (in \( L \)) and is thus accurately determined. An analogous argument shows that \( D \) and the bidiagonal decomposition of \( U \) are also accurately determined by \( b_{ij}, i > j \). □

**Proposition 2** Let \( Q \) be an orthogonal LTP matrix with LDU decomposition \( Q = LDU \). Let \( \bar{Q} \) be a perturbation of \( Q \) in the sense that \( \bar{Q} \) is orthogonal, its LDU decomposition is \( \bar{Q} = L\bar{D}U \) and the parameters \( b_{ij}, i > j \), in the bidiagonal decompositions of \( L \) and \( \bar{L} \) are identical with the exception of one entry \( \bar{b}_{ij} \), which is a small relative perturbation of the corresponding entry in the bidiagonal decomposition of \( \bar{L} \), i.e., \( \bar{b}_{ij} = b_{ij}(1 + \varepsilon) \), where \( \varepsilon \ll 1 \). Then the column vectors \( \bar{q}_i \) of \( \bar{Q} \) are small perturbations of the corresponding column vectors \( q_i \) of \( Q \):

\[
|q_i - \bar{q}_i| \leq 6n^3 \varepsilon.
\]

Make sure to read and refer to [19].

### 8 Appendix

In this appendix we give proofs of Theorem 1 and Lemma 6. The proofs are involved, but straightforward manipulations of the underlying bidiagonal decompositions.

Before we continue, we extend our notation from (1).

The matrices

\[
L^{(i)} \equiv \prod_{j=m-i+1}^{m} E_j(b_{j,i+m-j}) = \begin{bmatrix}
1 & & & & \\
 & \ddots & & & \\
 & & 1 & b_{i,m-i+1} & \\
 & & b_{i+1,m-i+2} & 1 & \\
& & & \ddots & \\
& & & b_{m,m-1} & 1
\end{bmatrix}
\]

are \( m \times m \) lower and \( n \times n \) upper bidiagonal, respectively. The decomposition (1) now becomes

\[
A = L^{(1)} \ldots L^{(n-1)} \cdot D \cdot U^{(n-1)} \ldots U^{(1)}.
\]

**Proof of Theorem 1**

We follow the construction of [4].

In the first part of the proof the authors construct unit lower triangular TN matrices \( R \) and \( P \) such that \( T = (RP^#)^{-1} \cdot A \cdot (RP^#) \) is tridiagonal.

The matrix \( R \) is constructed as a product of \( E_i(b_{ij}) \) with exactly one such factor for every entry in the lower triangular part that is set to zero. The order in which the zeros are created is the same as that of Neville elimination, thus

\[
R = \prod_{i=1}^{n-2} \prod_{j=n-i+1}^{n} E_j(b_{j,i+j-n}).
\]
Using the notation of (6) we have
\[ R = L^{(1)}L^{(2)} \cdots L^{(n-2)}, \]
where all nontrivial factors of \( L^{(1)}, L^{(2)}, \ldots, L^{(n-2)} \) are nonzero. Similarly we write
\[ P^\# = U^{(n-2)}U^{(n-1)} \cdots U^{(1)} \]
and thus
\[ V = L^{(1)}L^{(2)} \cdots L^{(n-2)}U^{(n-2)}U^{(n-1)} \cdots U^{(1)}. \]

We are clearly missing the last factor \( L^{(n-1)} \) in order to claim that the transformation matrix is LTP. This factor will come from the step of bidiagonalization of \( T \).

The construction from here on is involved but the idea is simple. The bidiagonalization of \( T \) consists of \( n-1 \) steps of updating \( V \) as
\[
V_i = V \cdot D \cdot E_n(1)E_{n-1} \cdots E_2(1)
\]
\[ = RP^\# \cdot D \cdot E_n(1)E_{n-1} \cdots E_2(1), \]
where \( D \) is diagonal. Using the techniques of [16, Section 4.2], we can propagate each of the factors \( D, E_n(1), E_{n-1}(1), \ldots, E_3(1) \) into the decomposition \( RP^\# \) changing the bidiagonal factors accordingly, but not affecting their nonzero structure. The last factor, \( E_2(1) \), we only propagate up to the left of the diagonal factor, obtaining
\[ V_i = L^{(1)}_i \cdot L^{(2)}_i \cdots L^{(n-2)}_iE_2(x_1)D_iU^{(n-2)}_iU^{(n-3)}_i \cdots U^{(1)}_i. \]

After \( n-1 \) such steps we have
\[ V_{n-1} = L^{(1)}_{n-1}L^{(2)}_{n-1} \cdots L^{(n-2)}_{n-1} \cdot (E_2(x_1)E_3(x_2) \cdots E_n(x_n)) \cdot D_{n-1}U^{(n-2)}_{n-1}U^{(n-3)}_{n-1} \cdots U^{(1)}_{n-1}. \]

Setting \( L^{(n-1)}_{n-1} = E_2(x_1)E_3(x_2) \cdots E_n(x_n) \) we obtain the decomposition
\[ V_{n-1} = L^{(1)}_{n-1}L^{(2)}_{n-1} \cdots L^{(n-1)}_{n-1}D_{n-1}U^{(n-2)}_{n-1}U^{(n-3)}_{n-1} \cdots U^{(1)}_{n-1} \]
making the similarity transformation matrix \( TN \) and LTP (but not TP).

**Proof of Lemma 6**

Let \( A = UDL \) be the UDL decomposition of \( A \). It suffices to prove that we can factor \( L \) as \( L = \tilde{L} \cdot E_i(x) \) for some \( x > 0 \), where \( \tilde{L} \) is again unit lower triangular and LTP.\(^3\)

We will use the reverse process of the one used in the last part of the proof of Theorem 4.3 in [16]; these transformations ((4.5)–(4.7) in [16]) produced the bidiagonal decomposition of the product of a lower triangular matrix and a matrix \( E_j(x) \). We are now in the reverse situation—“extracting” an \( E_j(x) \) from a lower triangular LTP matrix.

We start with the bidiagonal decomposition of \( L \)
\[ L = L^{(1)}L^{(2)} \cdots L^{(n-1)}. \]

Next, we split the \((n, n-1)\) entry in \( L^{(i-1)} \) in two and factor \( L^{(i-1)} \) accordingly. Let \( x_1 = \frac{1}{2}(L^{(i)})_{n,n-1} \). Then \( L^{(i-1)} = \tilde{L}^{(i-1)}E_n(x_1) \), where \( \tilde{L}^{(i-1)} \) equals \( L^{(i-1)} \) with the exception of its \((n, n-1)\) entry which is

\(^3\)The UDL decomposition of \( A \) and the bidiagonal decompositions of the factors \( L, D, \) and \( U \) in that decomposition can be easily obtained from the bidiagonal decomposition of \( A \) as follows. If the LDU decomposition of \( A^\# \) is \( A^\# = LDU \), then \( U = U^\#, \) \( D = D^\#, \) \( L = L^\#. \) The bidiagonal decompositions of converse matrices are easily obtained in a straightforward fashion as described in [17, section 5.4].
(\bar{L}^{(i-1)})_{n,n-1} = x_1. We move the “bulge” \( E_n(x_1) \) all the way to the right

\[
L = L^{(1)} L^{(2)} \cdots L^{(i-1)} \cdot L^{(i)} \cdot L^{(i+1)} \cdots L^{(n-1)} \\
= L^{(1)} L^{(2)} \cdots L^{(i-1)} \cdot E_n(x_1) \cdot \bar{L}^{(i)} \cdot L^{(i+1)} \cdots L^{(n-1)} \\
= L^{(1)} L^{(2)} \cdots L^{(i-1)} \cdot \bar{L}^{(i)} \cdot \bar{L}^{(i+1)} \cdot E_{n-1}(x_2) \cdot L^{(i+1)} \cdots L^{(n-1)} \\
= \cdots \\
= L^{(1)} L^{(2)} \cdots \bar{L}^{(i-1)} \cdot \bar{L}^{(i)} \cdot \bar{L}^{(i+1)} \cdots \bar{L}^{(n-1)} E_i(x_{n-i+1}) \\
= \bar{L} \cdot E_i(x_{n-i+1})
\]

(the matrices that are transformed on each step are underlined), where

\[ \bar{L} \equiv L^{(1)} L^{(2)} \cdots \bar{L}^{(i-1)} \cdot \bar{L}^{(i)} \cdot \bar{L}^{(i+1)} \cdots \bar{L}^{(n-1)}. \]

Each transformation step in (7) is performed using the relationship

\[ E_{n-j}(x_{j+1}) L^{(i+j)} = \bar{L}^{(i+j)} E_{n-j-1}(x_{j+2}) \]  \hspace{1cm} (8)

for \( j = 1, \ldots, n - i + 2. \)

Let the offdiagonal entries of \( L^{(i+j)} \) and \( \bar{L}^{(i+j)} \) in (8) be \( l_1, \ldots, l_{n-1} \) and \( \bar{l}_1, \ldots, \bar{l}_{n-1} \), respectively. Then by comparing entries of both sides of (8) we have that \( L^{(i+j)} \) equals \( \bar{L}^{(i+j)} \) with the exception of:

\[ \bar{l}_{n-j-1} = l_{n-j-1} + x_{j+1}, \quad \bar{l}_{n-j-2} = \frac{l_{n-j-2} l_{n-j-1} - l_{n-j-1}^2}{l_{n-j-1}}, \quad x_{j+2} = \frac{l_{n-j-2} x_{j+1}}{l_{n-j-1}}, \]

(this is analogous to (4.13)–(4.16) in [16]).

The key observation here is that the nonzero patterns of \( L^{(i+j)} \) and \( \bar{L}^{(i+j)} \) are the same, the nontrivial entries of \( \bar{L}^{(i+j)} \) remain positive, and \( x_{j+2} > 0 \).

Therefore at the end \( \bar{L} \) is LTP and \( x \equiv x_{n-i+1} > 0. \)

Note that this method is particularly fit for a numerical computation. Since it does not involve subtractions, it will not suffer from roundoff-induced subtractive cancellation and loss of relative accuracy.

References


