19. Hermitian inner products

Definition 19.1. *Let V be a complex vector space. A Hermitian inner product on V is a function*

$$\langle\ ,\ \rangle \colon V \times V \longrightarrow \mathbb{C},$$

which is

- ***conjugate-symmetric**, that is*
$$\langle u, v\rangle = \overline{\langle v, u\rangle}.$$
- ***sesquilinear**, that is linear in the first factor*
$$\langle \lambda u, v\rangle = \lambda\langle u, v\rangle,$$
for all scalars λ and
$$\langle u_1 + u_2, v\rangle = \langle u_1, v\rangle + \langle u_2, v\rangle,$$
for all vectors u_1, u_2 and v.
- ***positive** that is*
$$\langle v, v\rangle \geq 0.$$
- ***non-degenerate** that is if*
$$\langle u, v\rangle = 0$$
for every $v \in V$ then $u = 0$.

*We say that V is a **complex inner product space**. The **associated quadratic form** is the function*

$$Q \colon V \longrightarrow \mathbb{C},$$

defined by

$$Q(v) = \langle v, v\rangle.$$

Since a Hermitian inner product is linear in the first variable and conjugate-symmetric (so that switching factors we get the complex conjugate), we have

$$\langle u, \lambda v\rangle = \overline{\lambda}\langle u, v\rangle \qquad \text{and} \qquad \langle u, v_1 + v_2\rangle = \langle u, v_1\rangle + \langle u, v_2\rangle.$$

Note that

$$\langle v, v\rangle,$$

is a real number (it is equal to its complex conjugate, by conjugate-symmetry) and so it makes sense to ask for it to be non-negative. One can define the associated norm, as in the real case, and one can recover the Hermitian inner product from the norm, as in the real case.

The classic example of a Hermitian inner product space is the standard one on $\mathbb{C}^n$,

$$\langle x, y\rangle = \sum x_i \overline{y}_i.$$

For this inner product, we have

$$\langle Au, v\rangle = \langle u, \overline{A}^t v\rangle.$$

Definition 19.2. *Let $A \in M_{n,n}(\mathbb{C})$. We say that A is* ***Hermitian*** *if A is invertible and*

$$A^{-1} = \overline{A}^t.$$

Note that a real orthogonal matrix is Hermitian if and only if it is orthogonal.

Theorem 19.3 (Spectral Theorem)**.** *Let $A \in \mathbb{C}$ be a Hermitian symmetric matrix, so that*

$$\overline{A}^t = A.$$

Then A is diagonalisable and the eigenvalues of A are real.

Proof. Let $m(x)$ be the minimal polynomial of A. Then $m(x)$ has at least one complex root λ. But the roots of $m(x)$ are the eigenvalues of A and so A must have an eigenvector v with eigenvalue λ. Possibly rescaling, we may assume that the norm of v is one.

If we extend v to a basis of $\mathbb{C}^n$ and apply Gram-Schmidt (the same algorithm works for a Hermitian inner product) we may find an orthonormal basis of $\mathbb{C}^n$. Let V_2 be the span of the last $n-1$ vectors. Then V_2 is isomorphic to $\mathbb{C}^{n-1}$ with the standard Hermitian inner product and the restriction of A to V_2 defines a Hermitian matrix A_2 on $\mathbb{C}^{n-1}$. By induction on the dimension, A_2 has a basis of eigenvectors, and these gave a basis of eigenvectors of $\mathbb{C}^n$. Thus A is diagonalisable.

On the other hand

$$\begin{aligned}
\lambda\langle v, v\rangle &= \langle \lambda v, v\rangle \\
&= \langle Av, v\rangle \\
&= \langle v, \overline{A}^t v\rangle \\
&= \langle v, Av\rangle \\
&= \langle v, \lambda v\rangle \\
&= \overline{\lambda}\langle v, v\rangle.
\end{aligned}$$

Since $\langle v, v\rangle \neq 0$ it follows that $\lambda = \overline{\lambda}$ so that λ is real. It follows that the eigenvalues of A are real. □

Corollary 19.4. *Let $A \in M_{n,n}(\mathbb{R})$ be a symmetric matrix.*

Then A is diagonalisable.

Proof. Since A is real symmetric it is Hermitian. But then it is diagonalisable over $\mathbb{C}$. It follows that the sum of the dimensions of the eigenspaces is equal to n. As the eigenvalues of A are real and the nullity of A over $\mathbb{R}$ and over $\mathbb{C}$ are equal, it follows that the sum of the dimensions of the eigenspaces is equal to n, over $\mathbb{R}$. But then A is diagonalisable over $\mathbb{R}$. $\square$