Math 3A
Winter 2016
Final Exam
3/14/2016
Time Limit: 2 Hours

Name (Print): ______________________

Student ID ______________________

This exam contains 16 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	85	
2	50	
3	70	
4	50	
5	50	
6	50	
7	45	
Total:	400	

1. For $c \in \mathbb{R}$, we define the matrix $\mathbf{A_c} \in \mathbb{R}^{3\times 3}$ by

$$\mathbf{A_c} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 2 & 0 \\ 3 & c & 2 \end{bmatrix}. \tag{1}$$

(a) (10 points) Compute $det(\mathbf{A_c})$. Does it depend of c?

(b) (5 points) For which c is the matrix $\mathbf{A_c}$ invertible?

(c) (15 points) Compute $\mathbf{A_0}^{-1}$ (i.e. when $c = 0$).

(d) (5 points) Let $\mathbf{b} = (1, -4, 2)^t$, find the solution of $\mathbf{A_0 x} = \mathbf{b}$

(e) (5 points) Compute $det(\mathbf{A_c}^2)$.

(f) (5 points) Compute $det(5\mathbf{A_c})$.

(g) (5 points) Compute $det(\mathbf{E_k A_c})$, where

$$\mathbf{E}_k = \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

(h) (5 points) Compute $det(\mathbf{D_k A_c})$, where

$$\mathbf{D}_k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3}$$

(i) (5 points) Compute $det(\mathbf{A_0}^{-1})$.

(j) (15 points) Compute the eigenvalues of $\mathbf{A_0}$. Can you diagonalize $\mathbf{A_0}$?

(k) (10 points) Compute the eigenvalues of $\mathbf{A_0}^{-1}$.

2. Consider the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(\mathbf{x}) = \begin{pmatrix} 2x_1 + x_2 + \alpha x_3^2 \\ x_1 + 2x_2 \\ hx_3 + q \end{pmatrix}, \text{ where } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \tag{4}$$

in which, h and q are real numbers.

(a) (10 points) What are the conditions on q and α such that the transformation is linear?

(b) (10 points) **From now we suppose that** $q = 0$ **and** $\alpha = 0$. Write the associated matrix $\mathbf{A}$ of the transformation T. (**Hint:** remember that $\mathbf{A}(:, i) = T(\mathbf{e}_i)$.)

(c) (10 points) What is the condition on h such that the transformation T is **NOT** one-to-one? Explain briefly.

(d) (10 points) What is the condition on h such that the transformation T is **NOT** on to $\mathbb{R}^3$? Explain briefly.

(e) (10 points) **Suppose that** $h = 0$, then

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5}$$

Compute the eigenvalues of $\mathbf{A}$.

3. Let $\mathcal{B}$ be the set given by

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

and the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ given by

$$T \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad T \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \text{and } Nul(T) = Col(T) \tag{6}$$

(a) (15 points) Show that the $\mathcal{B}$ is a basis of $\mathbb{R}^4$.

(b) (15 points) Compute the standard (or associated) matrix of T using the canonical basis.

(c) (10 points) Show that a basis of $Nul(T)$ is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

(d) (10 points) Compute the standard (or associated) matrix of T using $\mathcal{B}$ as basis in the domain and codomain.

(e) (20 points) Compute the eigenvalues of the standard matrix of T. Is T diagonalizable?

4. Let S be the subspace given by

$$S = \text{span}\left\langle \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \right\rangle$$

(a) (10 points) Find a basis of S.

(b) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation such that $Nul(T) = S$ and

$$T \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad T \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}. \tag{7}$$

(a) (10 points) Find the rank of T.

(b) (15 points) Find a basis of $Col(T)$.

(c) (15 points) Provide an explicit formula for T.

5. Let $h \in \mathbb{R}$; Define

$$\mathbf{B}_h = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 - 10h \end{bmatrix}. \tag{8}$$

(a) (15 points) Compute a basis of Col $\mathbf{B}_0$.

(b) (15 points) Compute a basis of Nul $\mathbf{B}_0$.

(c) (10 points) Find the rank of $\mathbf{B}_0$ and the dimension of Nul $\mathbf{B}_0$.

(d) (10 points) For which values of h the rank of $\mathbf{B}_h$ is 3?

6. Consider

$$\mathbf{A} = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}. \quad (9)$$

(a) (10 points) Find the eigenvalues of $\mathbf{A}$ (**Hint**: use an educated guess)

(b) (10 points) Can you diagonalize $\mathbf{A}$? Explain briefly.

(c) (30 points) Compute a basis of $\mathbb{R}^4$ given by eigenvector of $\mathbf{A}$.

7. Let $B \in R^{n\times n}$ invertible such that $B^3 = 0$. For every $\lambda \in \mathbb{R}$ we define the matrix $M(\lambda) \in \mathbb{R}^n$ given by

$$M(\lambda) = I_n + \lambda B + \frac{\lambda^2}{2} B^2.$$

(a) (20 points) Show that

$$\forall \lambda, \beta \in \mathbb{R} \qquad M(\lambda + \beta) = M(\lambda) \cdot M(\beta),$$

conclude that $M(\lambda) \cdot M(\beta) = M(\beta) \cdot M(\lambda)$

(b) (25 points) Show that $M(\lambda)$ is invertible and that $M(\lambda)^{-1} = M(-\lambda)$ (**Hint**: use $M(0)$)