Question 1

a) $$\frac{d}{dx}\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x}}}} = \frac{\dfrac{\dfrac{\dfrac{1}{2\sqrt{x}}+1}{2\sqrt{x+\sqrt{x}}}+1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}+1}{2\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x}}}}}$$

b) $$\frac{d}{dx} x^2 \ln\left(\frac{\sin(x)}{x}\right) = x^3 \csc(x) \cdot \left[\frac{\cos(x)}{x} - \frac{\sin(x)}{x^3}\right] + 2x \ln\left(\frac{\sin(x)}{x}\right).$$

c) $$\frac{d}{dx}\left[(x^2+2)^2(x^4+4)^4\right] = 4x(x^2+2)(x^4+4)^3(5x^4+6x^2+4)$$

d) $$\frac{d}{dx} x^{\sin(x)} = x^{\sin(x)-1} \cdot (\sin(x) + x\log(x)\cos(x)).$$

e) $$\frac{d}{dx}(\sin(x))^{\ln(x)} = (\sin(x))^{\ln(x)}\left[\frac{\ln(\sin(x))}{x} + \ln(x)\cot(x)\right]$$

f) $$\frac{d}{dx}\left(\frac{\ln(x)}{1+\ln(x)}\right) = \frac{1}{x(\log(x)+1)^2}$$

g) $$\frac{d}{dx}\left((\ln(x))^{\cos(x)}\right) = (\ln(x))^{\cos(x)} \cdot \left(\frac{\cos(x)}{x\ln(x)} - \ln(\ln(x))\sin(x)\right)$$

h) $\frac{d}{dx}(\tan(x))^{1/x} = (\tan(x))^{1/x} \cdot \left[\frac{\csc(x) \cdot \sec(x)}{x} - \frac{\ln(\tan(x))}{x^2} \right]$

i) $\frac{d}{dx}\left(\sin(\cos(\arctan(x)))\right) = -\frac{x \cos\left(\frac{1}{\sqrt{x^2+1}}\right)}{(x^2+1)^{3/2}}$

j) $\frac{d}{dx} \sin(\ln(x)) = \frac{\cos(\ln(x))}{x}$

k) $\frac{d}{dx} \arcsin\left(\sqrt{\sin(x)}\right) = \frac{\cos(x)}{2\sqrt{1-\sin(x)} \cdot \sqrt{\sin(x)}}$

l) $\frac{d}{dx} \sqrt{\frac{1+x}{1-x}} = \frac{1}{(1-x)^2 \sqrt{\frac{1+x}{1-x}}}$

m) $\frac{d}{dx} \arctan\left(\sqrt{\frac{1-\cos(x)}{1+\cos(x)}}\right) = \frac{\frac{\sin(x)}{\cos(x)+1} + \frac{\sin(x)(1-\cos(x))}{(\cos(x)+1)^2}}{2\sqrt{\frac{1-\cos(x)}{1+\cos(x)}} \cdot \left(\frac{1-\cos(x)}{1+\cos(x)} + 1\right)}$

n) $\frac{d}{dx}\left(\ln\left(\frac{1+\sqrt{2} \cdot x + x^2}{1-\sqrt{2}x + x^2}\right)\right) = \frac{2\sqrt{2} \cdot (x^2-1)}{x^4+1}$

Question 2

Let's compute some derivatives of

$$f(x) = \ln(x-1)$$

so we can see if some kind of pattern appears

$$f'(x) = \frac{1}{x-1}$$

$$f''(x) = -\frac{1}{(x-1)^2}$$

$$f'''(x) = +\frac{2}{(x-1)^3}$$

$$f^{(4)}(x) = -\frac{2\cdot 3}{(x-1)^4}$$

$$f^{(5)}(x) = \frac{2\cdot 3\cdot 4}{(x-1)^5}$$

$$f^{(6)}(x) = -\frac{2\cdot 3\cdot 4\cdot 5}{(x-1)^6}$$

Then we can observe that each time we compute an extra derivative we multiply by $\frac{1}{(x-1)}$, we change the sign, and we multiply by $(n-1)$.

we have that

$$f^{(n)} = -\frac{(n-1)}{(x-1)}\cdot f^{(n-1)}$$

and $f^{(1)} = f' = \frac{1}{(x-1)}$

we can unroll the recursion and we have that

$$f^{(n)} = -\frac{(n-1)}{(x-1)}\cdot f^{(n-1)} = \frac{-(n-1)}{(x-1)}\cdot\left((-1)\cdot\frac{(n-2)}{x-1}\, f^{(n-2)}\right.$$

then we have

$$f^{(n)}(x) = \underbrace{\frac{(-1)(n-1)}{(x-1)} \cdot \frac{(-1)(n-2)}{(x-1)} \cdot \frac{(-1)(n-3)}{(x-1)} \cdots \frac{(-1)\cdot 2}{(x-1)}}_{n-1 \text{ times}} \cdot f^{(1)}$$

$$= \frac{(-1)^{n-1}\cdot(n-1)!}{(x-1)^{n-1}} \cdot f^{(1)} = \qquad \text{but } f^{(1)} = \frac{1}{(x-1)}$$

$$= \frac{(-1)^{n-1}(n-1)!}{(x-1)^{n-1}} \cdot \frac{1}{(x-1)} = \frac{(-1)^{n-1}(n-1)!}{(x-1)^{n}}$$

Then the formula is given by

$$\boxed{f^{(n)}(x) = \frac{(-1)^{n-1}\cdot(n-1)!}{(x+1)^{n}}.}$$

Question 3

We follow the hint:

$$f'(x) = \frac{d}{dx}\left(x e^x\right)$$ / product rule.

$$= \frac{d}{dx}x \cdot e^x + x \cdot \frac{d}{dx}e^x$$

$$= e^x + x e^x$$ / by definition of f.

$$= e^x + f$$

we have then

$$f' = e^x + f$$

if we compute the derivative at both sides of the equation we have.

$$f'' = e^x + f'$$

then we can compute any higher order derivate and we have

$$f^{(n)} = e^x + f^{(n-1)}$$

now we can unroll the recursion

$$f^{(n)} = e^x + f^{(n-1)} = e^x + e^x + f^{(n-2)} = e^x + e^x + e^x + f^{(n-3)}$$

we have that.

$$f^{(n)} = \underbrace{e^x + e^x + e^x \ldots + e^x}_{n \text{ times}} + f^{(0)}(x)$$

but $f^{(0)}(x) = f(x) = x e^x$

Thus $\boxed{f^{(n)} = ne^x + xe^x}$

Question 4

a) We want to show that

$$f = \cos(k g(t))$$

satisfies

$$f'' - f' \frac{g''}{g'} + (kg')^2 f = 0$$

to do so, we compute f'' and f'

$$\frac{d}{dx} f(x) = -\sin(kg(t)) \cdot kg'(t)$$ by chain rule.

$$\frac{d^2}{dx^2} f(x) = \frac{d}{dx}\left(\frac{d}{dx} f\right) = \frac{d}{dx}\Big(-\sin(kg(t)) \cdot kg'(t)\Big)$$ } product rule.

$$= \frac{d}{dx}\Big(-\sin(kg(t))\Big) \cdot kg'(t) + -\sin(kg(t)) \cdot \frac{d}{dx}(kg'(t))$$

$$= \Big(-\cos(kg(t)) \cdot kg'(t)\Big) \cdot kg'(t) + -\sin(kg(t)) \cdot kg''(t)$$

$$f'' = -\Big(\cos(kg(t)) \cdot (kg'(t))^2 + \sin(kg(t)) \cdot kg''(t)\Big)$$

Now we put every thing together.

$$\underbrace{-\Big(\cancel{\cos(kg)(kg')^2} + \sin(kg)\,\cancel{kg''(t)}\Big)}_{f''} - \underbrace{\Big(\cancel{-\sin(kg) \cdot kg'}\Big)}_{f'} \frac{g''}{g'}$$

$$+ \cancel{(kg')^2 \underbrace{\cos(kg)}_{f}} = 0$$

clearly everything cancels

b) if ~~x~~ $g(t) = t$. $g'(t) = 1$ $g''(t) = 0$

then f satisfies

$$f'' - f' \frac{0}{1} + (k \cdot 1)^2 f = 0$$

or $$f'' + k^2 f = 0.$$

$$f'' = -k^2 f$$

we can compute high-order derivatives and we obtain

$$f^{(n)} = -k^2 f^{(n-2)}.$$

We just need to evaluate

$f(0)$ and $f'(0)$ all the others can be deduced using the relation above.

if n is even.

then $$f^{(n)}(0) = -k^2 f^{(n-2)}(0) = -k^2(-k^2) \cdot f^{(n-4)}(0)$$
$$= \underbrace{(-k^2)(-k^2) \ldots (-k^2)}_{n/2 \text{ times.}} \cdot f^{(n-n)}(0)$$
$$= (-k^2)^{n/2} \cdot \underbrace{f^{0}(0)}_{=1} = (-k^2)^{n/2}.$$

on the other hand

if n is odd

$$f^{(n)}(0) = -k^2 f^{(n-1)}(0) = \underbrace{(-k^2)\dots(-k)^2}_{\frac{n-1}{2} \text{ times}} \cdot f'(0)$$

However $f'(0) = -\sin(k0)\cdot k = 0$

then

$$f^{(n)}(0) = 0 \quad \text{for } n \text{ odd.}$$

We can summarize as follows

$$\boxed{f^{(n)}(0) = \begin{cases} (-k^2)^{n/2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}}$$

Question 5

b) Let $f(x) = a^x = e^{(\ln a) \cdot x}$

then $f'(x) = e^{\ln(a) x} \cdot \ln(a)$.

however by definition $f = e^{\ln(a) \cdot x}$
which is equal to a^x
then we have that.

$$f'(x) = f(x) \ln(a)$$

Now we can compute high order derivatives using the equation we just found.

$$f^{(n)}(x) = \ln(a) \cdot f^{(n-1)}(x)$$

as usual we unroll the recursion

$$f^{(n)}(x) = \ln(a) \underbrace{f^{(n-1)}(x)}_{= \ln(a) f^{(n-2)}(x)} = \ln(a) \cdot \left[\ln(a) \cdot \underbrace{f^{(n-2)}(x)}_{\ln(a) \cdot f^{(n-3)}(x)} \right]$$

$$= \underbrace{\ln(a) \cdot \ln(a) \ldots \ln(a)}_{n \text{ times}} \cdot \underbrace{f^{(n-n)}(x)}_{f^{(0)}(x) = f(x) = a^x}$$

Then we have that

$$\boxed{\begin{aligned} f^{(n)}(x) &= (\ln(a))^n \cdot e^{\ln(a)\cdot x} \\ &= (\ln(a))^n \cdot a^x \end{aligned}}$$

Question 5

d) $$f(x) = \frac{1-x}{1+x}$$

we compute some derivatives to see if we can find a pattern.

$$f'(x) = \frac{(-1)(1+x) - (1-x)\cdot 1}{(1+x)^2}$$

$$= \frac{-1-x-1+x}{(1+x)^2} = \frac{-2}{(1+x)^2}.$$

then $$f''(x) = -2\,\frac{(-2)}{(1+x)^3}$$

$$f'''(x) = -2\,\frac{(-2)\cdot(-3)}{(1+x)^4}$$

Question 5

e) $f(x) = \sin^2(x)$

we may need to compute some derivatives in order to find a pattern.

$$f'(x) = 2\sin(x)\cdot\cos(x).$$

However, this is much simpler than we expected!

$$f'(x) = 2\sin(x)\cdot\cos(x) = \sin(2x).$$

then we can just part 1 to conclude:

$$\boxed{f^{(n)}(x) = \begin{cases} -2^{n-1}\cos(2x) & \text{if } n = 4k \\ 2^{n-1}\sin(2x) & \text{if } n = 4k+1 \\ 2^{n-1}\cos(2x) & \text{if } n = 4k+2 \\ -2^{n-1}\sin(2x) & \text{if } n = 4k+3 \end{cases}}$$

Question 6

a) Let $f(x) = x^2 - \cos(x)$

Clearly f is continuous in $\mathbb{R}$.
moreover let's consider the interval $[-\frac{\pi}{2}, 0]$ clearly

$$f(0) = 0^2 - \cos(0) = -1 < 0$$

and $f(\frac{-\pi}{2}) = \left(\frac{-\pi}{2}\right)^2 - \cos\left(\frac{-\pi}{2}\right) = \left(\frac{-\pi}{2}\right)^2 - 0 = \frac{\pi^2}{4} > 0.$

Then we can use the intermediate value. Theorem to show that $\exists\, c \in (-\frac{\pi}{2}, 0)$ such that $f(c) = 0$

We have found one zero. We can easily find another zero by considering the interval $[0, \frac{\pi}{2}]$

To check that we only have two zeros we need to study the derivative of f

clearly $f'(x) = 2x + \sin(x)$.

and the only solution to $f'(x) = 0$

is $x = 0$.

Now if $x > 0$ then $f'(x) > 0$.

this is because $x > \sin(x)$.

so even if $\sin(x)$ changes sign $f'(x) > 0$.

This means that $f(x)$ always grow as $x \to +\infty$; Therefore, given that f is continuous. it will only intersect once the line $y = 0$.

Using the same argument.

if $x < 0$ then $f'(x) < 0$. This means that f is always decreasing in the interval $(-\infty, 0]$. Therefore it will intersect only one the line $y = 0$.

Then we can conclude that $f(x)$ has only two zeros $\boxed{}$

b) Let $g(x) = 2x^2 - x\sin(x) - \cos^2(x)$.

g is clearly continuous in $\mathbb{R}$. Then we will follow the same procedure as before.

first $g(0) = 2 \cdot 0^2 - 0 \cdot \sin(0) - \cos^2(0)$

$= -1 < 0$.

now we need to find two numbers (one at the left of 0, and one at the right) such that $g(x) > 0$ so we can use the intermediate value Theorem.

Clearly for $x = -\pi$ $\quad g(-\pi) = 2 \cdot (-\pi)^2 - (-\pi)\sin(-\pi) - \cos^2(-\pi)$

$= 2\pi^2 - 1 > 0$.

and for $x = \pi$ $\quad g(\pi) = 2(\pi)^2 - \pi\sin(\pi) - \cos^2(\pi)$

$= 2\pi^2 - 1 > 0$.

Now, by the IVT we have tha g has at least two zeros one in the interval $(-\pi, 0)$ and the other in $(0, \pi)$.

Finally, we need to show that they are unique.

Once again we study the derivative of $g(x)$.

$$g'(x) = 4x - x\cos(x) - \sin(x) + 2\cos(x)\sin(x).$$

we have that $g'(0) = 0$.

Moreover, we know that $2\cos(x)\sin(x)$ is equal to $\sin(2x)$.

then

$$g'(x) = 4x - x\cos(x) - \sin(x) + \sin(2x)$$

if $x>0$ then $x > \sin(x)$ seen in class!
$2x > \sin(2x)$.

and $x \geq x\cos(x)$. because $-1 \leq \cos(x) \leq 1$

then we have that $x>0$. $g'(x) > 0$.

then it will only intersect once the line $y=0$. in the interval $(0, +\infty)$

for $x<$ the same argument holds, which allows us to conclude ☒

Question 8

$$g(x) + x\sin(g(x)) = x^2$$

then we use implicit differentiation and the chain rule.

$$g'(x) + \sin(g(x)) + x\cos(g(x))\cdot g'(x) = 2x.$$

then $$g'(x)\left[1 + x\cos(g(x))\right] = 2x - \sin(g(x)).$$

and
$$g'(x) = \frac{2x - \sin(g(x))}{1 + x\cos(g(x))}.$$

now we just need to know the value of g at 0. For that we just use the definition of g.

$$g(0) + 0\cdot\sin(g(0)) = 0.$$

$$\Rightarrow \quad \boxed{g(0) = 0}$$

then by substitution in $g'(x)$

$$g'(0) = \frac{2\cdot 0 - \sin(0)}{1 + 0\cdot\cos(0)} = 0. \qquad \boxed{g'(0) = 0}$$

Question 9/.

if $f(x) = (x-a)(x-b)(x-c)$

then $f'(x) = (x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)$

using the product rule.

Then we just need to replace the expression of f' in

$$\frac{f'}{f} = \frac{(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)}{(x-a)(x-b)(x-c)}.$$

then we have.

$$\frac{f'}{f} = \frac{\cancel{(x-b)}\cancel{(x-c)}}{(x-a)\cancel{(x-b)}\cancel{(x-c)}} + \frac{\cancel{(x-a)}\cancel{(x-c)}}{\cancel{(x-a)}(x-b)\cancel{(x-c)}} + \frac{\cancel{(x-a)}\cancel{(x-b)}}{\cancel{(x-a)}\cancel{(x-b)}(x-c)}$$

$$= \frac{1}{(x-a)} + \frac{1}{(x-b)} + \frac{1}{(x-c)}$$

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Question 10

compute the limit

$$\lim_{x\to 0} x^2 \ln\left(\frac{\sin(x)}{x}\right)$$

now $\displaystyle\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$

and $\ln(x)$ is continuous and well defined at 1; and x^2 is clearly continuous.

Then by continuity.

$$\lim_{x\to 0} x^2 \ln\left(\frac{\sin(x)}{x}\right) = \underbrace{\underbrace{\left(\lim_{x\to 0} x\right)^2}_{=0} \cdot \ln\left(\underbrace{\lim_{x\to 0} \frac{\sin(x)}{x}}_{=1}\right)}_{=0}.$$

$$= 0$$