The Polynomial Method, Fall 2012, Problem Set 3

Due on Monday Nov. 19 in class.

1. Reconstruct the proof of the crossing number theorem and the Szemerédi-Trotter theorem from your outline on the last problem set.

2. Summarize the proofs for the 3-rich point bound (Lecture 15) and the 3-dimensional incidence bound for lines in space with not too many lines in a plane (Lecture 20). These proofs are getting longer. Obviously there's a trade-off between having a short summary and including useful information for reconstructing the proof. I think a summary with 4-6 steps is probably good for these.

3. Behrend example. Let n be an integer. For any $R \geq 10$, find an integer $A \leq R^2$ so that the equation $\sum_{i=1}^n x_i^2 = A$ has $\gtrsim R^{n-2}$ integer solutions with $|x_i| \leq R$ for each i. Let $S_0 \subset \mathbb{Z}^n$ be the set of solutions of $\sum_{i=1}^n x_i^2 = A$.

Prove that S_0 has no three term arithmetic progression. (A 3-term arithmetic progression in $\mathbb{Z}^n$ is a sequence $a, a+d, a+2d$, with $a, d \in \mathbb{Z}^n$.)

Let Q_R^n denote the cube of integer points $x \in \mathbb{Z}^n$ with $|x_i| \leq R$. Notice that $S_0 \subset Q_R^n$, and $|S_0| \gtrsim |Q_R^n|^{\frac{n-2}{n}}$. (You don't have to write anything for this step.)

Find a nice map from Q_R^n to Q_N^1 for some N, and let S be the image of S_0. Prove that S has no 3-term arithmetic progression and that $|S| \gtrsim N^{\frac{n-2}{n}}$.

4. Incidences of algebraic curves in the plane. Suppose that $\mathfrak{L}$ is a set of L irreducible degree d curves in $\mathbb{R}^2$, and $\mathfrak{S}$ is a set of S points in $\mathbb{R}^2$. We write $A \lesssim B$ for $A \leq C(d)B$.

a.) Just by counting prove the following estimates on the number of incidences:

- $|I(\mathfrak{S}, \mathfrak{L})| \lesssim L + S^{d^2+1}$.
- $|I(\mathfrak{S}, \mathfrak{L})| \lesssim L^2 + S$.

b.) Using a polynomial cell decomposition, prove a better estimate on the number of incidences. Suppose $\mathfrak{L}$ is a set of L irreducible degree d curves in the plane, and let $\mathfrak{S}_r$ be the set of points in $\geq r$ curves of $\mathfrak{L}$. Using your incidence bound prove that

$$|\mathfrak{S}_r| \lesssim L^2 r^{-2-\frac{1}{d^2}} + L r^{-1}.$$

c.) (optional) What's the best example you can find with $d = 2$?

Remarks. This theorem was first proven by Pach and Sharir using the crossing number theorem.

5. Suppose that $\mathfrak{L}$ is a set of N^2 lines in $\mathbb{R}^3$, with $\leq N$ lines in any plane. Suppose that $\mathfrak{S}$ is a set of points containing at least N points on each line of $\mathfrak{L}$. Prove that $|\mathfrak{S}| \gtrsim N^3$.