The distinct distance problem and the unit distance problem

During this lecture, we examine the distinct distance problem and the unit distance problem. We would like to apply the following theorem about crossing numbers, proven last time:

Theorem 1. *If G is a graph with E edges and V vertices, and $E \geq 4V$, then the crossing number $K(G)$ of G is at least $(1/64)E^3V^{-2}$.*

1 Distinct distance problem

Suppose we have a set S of N points in the plane, and amongst them we have t distinct distances, $t < N$. How small can t be?

Let T be the set of distances that arise between two points. We can draw Nt circles: for each point $p \in S$ and each distance $d \in T$, we draw a circle with center p and radius d. We construct a graph G: the vertices will be the points of S, and the edges will be the arcs of circles between consecutive points. What is the crossing number of the graph? We know that any two circles intersect in at most two points. So we have the inequality $K(G) \leq 2\binom{Nt}{2} \leq (Nt)^2$. On the other hand, we know that the graph has N points. Each point is contained in $N-1$ circles, since all distances that arise between points is contained in T, so each point has degree $2(N-1)$. So, there are $N(N-1)$ edges. So, we have the following inequality:

$$N^2t^2 \geq K(G) \geq 1/64E^3V^{-2} = 1/64N^3(N-1)^3N^{-2} \geq 1/100N^4$$

This gives $t \geq 1/10N$.

This proof, however, is incorrect. When we proved the theorem for crossing numbers, we assumed that the graph was simple, however, the graph can have both multiple edges and loops: If a circle has a single point, that point will have a loop around it, and if there are two points P, Q, and many other points on their perpendicular bisector, then P, Q will have many circles going through them, and if there are no interior points on the arcs, there can be many edges between P and Q.

Could the crossing numbers theorem work for graphs that are not simple? In the proof, we used the fact that $3F \geq 2E$. However, this is only true because each face has at least three incident edges. If we allow multiple edges, this is not necessarily the case. So our proof from last time does not work.

In fact, if we have a planar graph, we can take an edge, and draw as many parallel edges as we want, without obtaining any new crossing. So the theorem is false for non-simple graphs, and the proof above in this form fails.

Can we try to obtain some sort of theorem on crossing numbers for graphs with multiple edges? Obviously not for general non-simple graphs, but perhaps with some conditions, we can.

Definition 1. *We will use the term* **multigraph** *to refer to a graph that can have multiple edges, but no loops. For a multigraph G, define $Mult(G)$ as the highest number of parallel edges between two points, so $Mult(G) \leq M$ implies that no two points have more than M edges between them.*

Proposition 1. *If $Mult(G) \leq M$, and $E \geq 4MV$, then $K(G) \geq 1/64E^3V^{-2}M^{-3}$.*

Proof. Take $G' \subset G$ to be the graph where we replace any parallel edges by one edge. If this graph has E' edges and V' vertices, then $V' = V$, and $E' \geq 1/ME$. Obviously, $K(G') \geq K(G)$, so we have

$$K(G') \geq K(G) \geq 1/64E'^3V'^{-2} \geq 1/64E^3V^{-2}M^{-3}$$

$\square$

From this, we can deduce the folloming:

Theorem 2. *If we have N points in the plane, no 100 of which are on a common line, then the number of distinct distances is at least cN, where c is a constant.*

Proof. Take the multigraph that is the graph we defined earlier, with the circles that have only one point on them omitted (since these would be loops). By removing these circles, we removed at most Nt edges, so if $t \leq N/2$, then there are still at least $1/2N^2$ edges. (Otherwise we are done.) Since there are less than 100 points on any line, we can have at most 200 edges between any two points. Thus, using the previous proposition, we obtain the desired result. $\square$

2 Unit distance problem

Last time, we constructed an example by taking a square $n \times n$ grid of the right size, a set of N points with $U(N)$ unit distances, where $\omega(N) \leq U(N) \leq O(N^{1+\epsilon})$ for any epsilon.

Using Proposition 1, we can deduce the following theorem:

Theorem 3. *A set S of N points in the plane determine at most $U \leq O(N^{4/3})$ unit distances.*

Proof. Draw all the unit circles that have centers in S, and contain at least two points of S, and take the multigraph as before: the points are S, and the edges are arcs along circles. Assume $U \geq 10N$ (otherwise we are done). If we first draw those unit circles that have at least one point on them, not just those with at least two, then we obtain a graph with at least $2U$ edges: given a pair of points P, Q that are a unit distance apart, look at the circle with center P. This goes through Q, and we can assign to this unit distance the arc of the circle with center P that is to the left of Q (looking at it from P), and vice versa. Thus, we have two edges for each unit distance. Now, if we delete those circles that have one point on them, we delete at most N circles, so we still have at least U edges. This graph has multiplicity at most 4: given any two points, there can be at most two unit circles that go through both of them, and each unit circle gives us at most two edges between them. Since any two circles intersect in at most two points, we can write $2N^2 \geq K(G) \geq cE^3V^{-2} \geq c'U^3N^{-2}$, which gives $U \leq O(N^{4/3})$ as required. $\square$

This result is almost 30 years old, published by Spencer, Szemerédi, and Trotter in 1984. This is the best known upper bound, to this day. One reason it is hard to improve is that it is hard to distinguish unit circles from unit parabolas. In that case, however, we have the following example:

Look at parabolas of the form $y = x^2 + ax + b$, let a run from 1 to s, let b run from 1 to s^2. Look at the grid of points in the plane $[1, 2, ..., s] \times [1, 2, ..., 3s^2]$, this gives us $3s^3$ points. Each of the parabolas defined above has s points on it, since plugging in any value of x from 1 to s gives a value of y from 1 to $3s^2$. Thus, this example would give us at least $s^4 = cN^{4/3}$ "unit distances", or incidences between unit parabolas and points.

Definition 2. *Given a set L of curves in the plane and another set S of points, we define the set of incidences $I(S, L) = \{(x, l) \in S \times L : x \in l\}$.*

If we could obtain a similar example for unit circles, then we could add the set of centers of circles, and obtain a counterexample. In fact, for different norms, we can obtain such a set in a similar way.

3 Crossing numbers for multigraphs

Let us return to the distinct distance problem, and the problem of crossing numbers for multigraphs.

What is the crossing number of K_5^M, that is, K_5 with each edge drawn M times?

We can easily embed it into the plain to get M^2 crossings: embed K_5 such that it has one crossing, and draw each edge M times. Can we do better?

Suppose we have an embedding of K_5^M with $K(K_5^M)$ crossings. Take a random subgraph G', where we randomly choose each edge from the M parallel edges. In the induced embedding on the subgraph, each crossing occurs with probability $1/M^2$, since it occurs if and only if both edges are in G'. Thus, the expected value of the number of crossings is $1/M^2 K(K_5^M)$, and so

$$1 \leq \mathbb{E}(K(G')) \leq K(K_5^M)/M^2$$

We can generalize this idea to obtain the following lemma, that gives a better bound than the proposition from earlier:

Lemma 1. *Let G be a multigraph with multiplicity at most M. Assume $E \geq 4MV$, and each edge has multiplicity greater than $M/2$. Then $K(G) \geq 1/256 E^3 V^{-2} M^{-1}$.*

So we can see that although the constant is a bit worse, we have M^{-1} instead of M^{-3}.

Proof. Take a random $G' \subset G$ subgraph, where we randomly choose one edge from each set of parallel edges. Since each set has at least $M/2$ edges, each crossing remains with probability at most $M^2/4$. Since the multiplicity is at most M, $E' \geq 1/ME \geq 4V$. Thus, we can write

$$4/M^2 K(G) \geq \mathbb{E}(K(G')) \geq 1/64(E')^3 V^{-2} \geq E^3 V^{-2} M^{-3}$$

$\square$

Using this lemma, we can prove the following proposition (where we get rid of the lower bound on edge multiplicities):

Proposition 2. *If G is a multigraph with multiplicity at most M, and $E \geq 100MV$, then $K(G) \geq cE^3 V^{-2} M^{-1}$ for some c.*

Proof. Let $G' \subset G$ consist of all edges that have multiplicity at least $M/2$.

If $E' \geq 1/10E \geq 10MV$, then, using the lemma,

$$K(G) \geq K(G') \geq 1/256 E'^3 V^{-2} M^{-1}) \geq cE^3 V^{-2} M^{-1}$$

If $E' < 1/10E$, then take G_1 to be the edges not in G, and use induction on G_1, with $M/2$ (this contains at least $9/10$ of the edges, and the multiplicity is halved, so the $E \geq 100MV$ condition is satisfied). So we have $E_1 \geq 9/10E$, and so

$$K(G) \geq K(G_1) \geq cE_1^3 V^{-2}(M/2)^{-1} \geq (9/10)^3 2cE^3 V^{-2} M^{-1} \geq cE^3 V^{-2} M^{-1}$$

$\square$