CROSSING NUMBERS AND THE SZEMERÉDI-TROTTER THEOREM

In this lecture we study the crossing numbers of graphs and apply the results to prove the Szemeredi-Trotter theorem. These ideas follow the paper "Crossing numbers and hard Erds problems in discrete geometry" by László A Székely (Combin. Probab. Comput. 6 (1997), no. 3, 353-358).

1. Crossing number estimates

Proposition 1.1. *If G is a planar graph with E edges and V vertices, then $E-3V \leq 0$.*

Proof. We can reduce to the case that G is connected.

Suppose that G is planar and consider an embedding of G into S^2. This embedding cuts S^2 into faces, and we get a polyhedral structure on S^2 with V vertices, E edges, and some number F of faces. By the Euler formula, $V - E + F = 2$. The number of faces cannot easily be read from the graph G, but we can estimate it as follows. Each face has at least three edges in its boundary, whereas each edge borders exactly two faces. Therefore $F \leq (2/3)E$. Plugging in we get

$$2 = V - E + F \leq V - (1/3)E.$$

Rearranging gives $E - 3V \leq -6$, and we're done. $\square$

Technical details: Why did we assume G connected? Consider a graph homeomorphic to two circles, embedded in S^2 as two concentric circles. This gives three "faces" - two disks and an annulus. The Euler formula is false for this configuration because annular faces are not allowed. In class, we discussed some other configurations that require thought, like a single edge, and a tree. There is an interesting book by Lakatos that describes of difficulty of correctly formulating the hypotheses of the Euler formula.

If $E-3V$ is positive, then we see that G is not planar, and if $E-3V$ is large then we may expect that G has a large crossing number. We prove a simple bound for this now.

Proposition 1.2. *The crossing number of G is at least $E - 3V$.*

Proof. Let $k(G)$ be the crossing number of G. Embed G in the plane with $k(G)$ crossings. By removing at most $k(G)$ edges, we get a planar graph G' with $E' = E-k$ edges and $V' \leq V$ vertices. We see $0 \geq E' - 3V' \geq E - k - 3V$. $\square$

For perspective, consider the complete graph K_n. It has n vertices and $\binom{n}{2}$ edges. For large n, this proposition shows that the crossing number of K_n is $\gtrsim n^2$. On the other hand, the only upper bound we have so far is the trivial bound that the crossing number of K_n is $\lesssim n^4$.

What may we hope to improve in this proposition? When we remove an edge of G, it's in our interest to remove the edge with the most crossings, and when we do this, the crossing number of G can decrease by more than 1. For example, for the complete graph K_n, it looks plausible that there is always an edge with $\sim n^2$ crossings. How may we estimate this?

This seems to be a tricky problem, and Székely found a very clever solution. Instead of trying to prove that one edge intersects many other edges, he considered a small random subgraph $G' \subset G$ and proved that two edges of G' must cross. Since G' is only a small piece of G, it follows that many pairs of edges in G must cross.

Theorem 1.3. *If G is a graph with E edges and V vertices, and $E \geq 4V$, then the crossing number of G is at least $(1/64)E^3V^{-2}$.*

This theorem was proven by several authors before Székely, but we give his proof. It shows that the crossing number of the complete graph K_n is $\gtrsim n^4$ as a special case.

Proof. Let p be a number between 0 and 1 which we choose below. Let G' be a random subgraph of G formed by including each vertex of G independently with probability p. We include an edge of G in G' if its endpoints are in G'.

We consider the expected values for the number of vertices and edges in G'. The expected value of V' is pV. The expected value of E' is p^2E. For every subgraph $G' \subset G$, the crossing number of G' is at least $E' - 3V'$. Therefore, the expected value of the crossing number of G' is at least $p^2E - 3pV$.

On the other hand, we give an upper bound on the expected crossing number of G' as follows. Let $k = k(G)$ be the crossing number of G. Let $F : G \to \mathbb{R}^2$ be a legal embedding with k crossings. We claim that each crossing of F involves two disjoint edges. In other words, two edges that share a vertex don't cross. We come back to the claim at the end. By restricting F to G', we get an embedding of G' with p^4k crossings on average. This is because each crossing involves four vertices, and it appears as a crossing of $F(G')$ only if all four vertices are included in G'. (If F had a crossing involving two edges containing a common vertex, then it would appear with the much higher probability p^3.) Therefore, the expected value of the crossing number of G' is at most p^4k.

Comparing our upper and lower bounds, we see that $p^4k \geq p^2E - 3pV$, and so we get the following lower bound for k.

$$k \geq p^{-2}E - 3p^{-3}V.$$

We can now choose p to optimize the right-hand side. We choose $p = 4V/E$, and we have $p \leq 1$ since we assumed $4V \leq E$. Plugging in we get $k \geq (1/64)E^3V{-}2$.

To finish the proof, we just have to check the claim that F has no crossings of edges that share a vertex. Given any map with such a crossing, we explain how to modify it to reduce the crossing number. Say that $F(e_1)$ and $F(e_2)$ each leave $F(v)$ and cross at x. (If they cross several times, then let x be the last crossing.) We modify F as follows. Suppose that $F(e_1)$ crosses k_1 other edges on the way from $F(v)$ to x and that $F(e_2)$ crosses k_2 other edges on the way from $F(v)$ to x. We choose the labelling so that $k_1 \leq k_2$. Then we modify F on the edge e_2, making $F(e_2)$ follow parallel to $F(e_1)$ until x and then rejoin its original course at x, so that $F(e_1)$ and $F(e_2)$ never cross. This operation reduces the crossing number of x, and so a minimal map F has no such crossings. □

2. The Szemerédi-Trotter theorem

Theorem 2.1. *Let $\mathfrak{L}$ be a set of L lines in the plane. Let P_k be the set of points that lie on at least k lines of $\mathfrak{L}$. Then the number of points in P_k is at most* $\max(2Lk^{-1}, 2^9L^2k^{-3})$.

Proof. Using the lines and points, we make a graph mapped into the plane. The vertices of our graph G are the points of P_k. We join two vertices with an edge of G if the two points are two consecutive points of P_k on a line $l \in \mathfrak{L}$. This graph is not embedded, but the crossing number of our map is at most $\binom{L}{2} \leq L^2$, since each crossing of the graph G must correspond to an intersection of two lines of $\mathfrak{L}$.

We will count the vertices and edges of the graph G and apply the crossing number theorem. The number of vertices of our graph is $V = |P_k|$. The number of edges of our graph is $kV - L$. (At first sight, each vertex should be adjacent to $2k$ edges which would give kV edges. But on each line $l \in \mathfrak{L}$, the first and last vertices are adjacent to one less edge than this initial count.) As long as $E \geq 4V$, we can apply the crossing number theorem and it gives

$$L^2 \geq (1/64)(kV - L)^3V^{-2}.$$

Either $V \leq 2L/k$, or else $kV - L \geq (1/2)kV$. In the former case, we are done. In the latter case, we have $L^2 \geq 2^{-9}k^3V$, which means $V \leq 2^9L^2k^{-3}$.

On the other hand, if $E < 4V$, we have $kV - L \leq 4V$, and hence $V \leq \frac{L}{k-4}$. As long as $k \geq 8$, this implies $V \leq 2L/k$, and we are done. Finally, for $k < 8$, the trivial bound $|P_k| \leq \binom{L}{2}/\binom{k}{2} \leq 2L^2k^{-2} \leq 2^9L^2k^{-3}$. □