Pesto

1

Fall 2012

1 Incidence Geometry

Topic: take a bunch of simple shapes like circles or lines, and study how they can intersect each other.

Definition 1.1. If L is a set of $|L|$ lines in $\mathbb{R}^2$, let $P_k(L)$ be the set of points lying in at least k lines, called k-fold intersections; then we can ask what the maximum value of $P_k(L)$ in terms of k and $|L|$ is.

For example, we can get $P_k(L) = |L|/k$ trivially by dividing the lines into sets of k and intersecting each set.

In an $N \times N$ grid of points, let L be the set of lines that contain between R and $2R$ points. Then there are at most $\theta(\frac{N^2}{R^2})$ lines of those lines through each point: in any such line, the closest point to x must lie in a square of sidelength $\frac{2N}{R}$ centered at x. We claim that there are at least $\theta(\frac{N^2}{R^2})$ of those lines through each point, too: each of the points in the quarter of that square of sidelength $\frac{2N}{R}$ closest to the center of the grid determines a line that contains at least R points, and by the following lemma, a constant fraction of them are distinct and contain not too many points:

Lemma 1. For all B, there are more than $\frac{1}{100}B^2$ integer pairs $(x, y) \in [\frac{B}{2}, \frac{B}{]}^2$ with gcd 1

Proof. Throw out the $\frac{1}{4}$ pairs where both are divisible by 2, the $\frac{1}{9}$ divisible by 3, and so on. $\frac{1}{4} + \frac{1}{9} + \cdots < \frac{99}{100}$. □

If k is the smallest degree of any grid point, then k is about $\frac{N^2}{R^2}$, $|P_k| \geq N^2$, and $|L| = |P_k|k/R = \frac{N^4}{R^3}$, so $|P_k| = |L|^2 k^{-3}$.

Proposition 1.2. *$\forall k \in [\sqrt{|L|}]$, there's a configuration such that $|P_k| \geq cL^2K^{-3}$.*

In the early 1980s, it was proven that one of the two bounds above is tight up to a constant factor:

Theorem 1.3 (Szemerédi, Trotter)**.** *For some constant c, $|P_k| \leq c\left(\frac{|L|}{k} + \frac{|L|^2}{k^3}\right)$.*

If $k > \sqrt{|L|}$, the first term dominates; if $k \geq \sqrt{L}$, the second term dominates.

Proposition 1.4. $|P_k| \leq \frac{\binom{|L|}{2}}{\binom{K}{2}} \leq 2L^2k^{-2}$

Proof. There are $\binom{L}{2}$ pairs of lines, and $\forall x \in P_k$, there are at least $\binom{k}{2}$ pairs of lines that intersect at x. □

Proposition 1.5. *Prop: If $\frac{k^2}{4} > |L|$, then $|P_K| < \frac{k}{2}$.*

Proof. Suppose not. Restrict to a subset P of size $\frac{k}{2}$. For all $x \in P$, there are at least $\frac{k}{2}$ lines through x that don't contain any other points of P, so $|L| \geq |P|\frac{k}{2} = \frac{k^2}{4}$. $\square$

Proposition 1.6. *If* $|L| < \frac{k^2}{4}$*, then* $|P_k| < 2\frac{|L|}{k}$.

Proof. Suppose not. By the last proposition, $|P_k| < \frac{k}{2}$. For all $x \in P$, there are at least $\frac{k}{2}$ lines through x that don't contain any other points of P, so $|L| \geq |P_k|\frac{k}{2}$, as desired. $\square$

So far, we've only used the fact that two lines intersect in at most one point. But that can't be enough to prove the Szemerédi-Trotter Theorem, because in a finite field $\mathbb{F}_q^2$, we could take *all* the lines: that gives $|L| = q^2 + q$ and $k = q + 1$, which violates the Szemerédi-Trotter upper bound. (Note that in that case there's a phase transition around $k = \sqrt{|L|}$, from $|P_k| = \sqrt{|L|}$ to $|P_k| = |L|$.)

The extra fact we'll use is some topology, specifically the Euler characteristic. Take a large disc containing all the intersections and let V_{int} and E_{int} be the interior vertices and edges; there are also $2|L$ vertices and $2|L|$ edges along the boundary of the disc. Every edge is in at most two faces (1 if along the boundary) and every face contains at least three edges, so $3|F| \leq 2|E_{int}| + 2|L|$, so $|E_{int}| \leq 3|V_{int}| + 2|L|$. Hence $\sum_{v \in V_{int}} (\frac{1}{2}\deg(v) - 3) \leq 2L$ (in fact, it's at most L). If every intersection had multiplicity at least 3, then $|P_k| \leq \frac{2L}{K-3}$; we need to figure out a stronger argument because intersections might have multiplicity 2.

K_5 isn't planar, since $10 = |E(K_5)| > 3|V(K_5)| - 6 = 9$.

2 Crossing Numbers of Graphs

If G is a graph, a legal map F into the plane takes vertices to distinct points and edges to curves between their endpoints' points.

The crossing number of F is the number of pairs of edges' curves that intersect, and the crossing number of a graph is the minimum crossing number over legal embeddings. For instance, $CN(K_5) = 1$.