THE JOINTS PROBLEM

Suppose that we have a set of lines in $\mathbb{R}^3$. A joint of the set of lines is a point which lies in three non-coplanar lines. The joints problem asks what is the maximal number of joints that can be formed from L lines.

Let's look at some examples.

1. A grid. An $S \times S \times S$ grid of lines contains $3S^2 = L$ lines and contains S^3 joints. So the number of joints is $\sim L^{3/2}$.

2a. A tetrahedron. Six lines arranged as the edges of a tetrahedron produce four joints. This example generalizes as follows.

2b. Take S planes in $\mathbb{R}^3$ in general position. Any two of the planes intersect in a line, giving $L = \binom{S}{2}$ lines. Any three of the planes intersect in a point, and each such point is a joint for this set of lines. Therefore, these lines determine $\binom{S}{3}$ joints. We again have that the number of joints is $\sim L^{3/2}$, but the constant is better than in 1. These are the best known arrangements of lines in the joints problem.

Next we look at upper bounds. Since any two lines intersect in ≤ 1 point, there are $\leq \binom{L}{2}$ intersection points, and so $\leq \binom{L}{2}$ joints. A joint involves not just two lines, but three lines. We remark that looking at triple intersections (intersections of three lines) is not very different from just intersections. It's possible to arrange L lines in a plane to give $\sim L^2$ triple intersections. Start with an evenly spaced grid of vertical and horizontal lines, and then add diagonal lines to make the triple intersections. But this example does not give any joints, because all the lines are coplanar.

The joints problem was posed in the early 90's by B Chazelle, H Edelsbrunner, L.J Guibas, R Pollack, R Seidel, M Sharir, and J Snoeyink, in the paper 'Counting and cutting cycles of lines and rods in space' (Comput. Geom. Theory Appls., 1 (1992), pp. 305323). They proved that the number of joints from L lines is $\lesssim L^{7/4}$, and the exponent has gradually improved. We will explain some of the ideas from that first paper later. We mention that it's not easy to prove that the number of joints is $\lesssim L^{1.99}$.

With the polynomial method, we now have a rather sharp bound for the joints problem.

Theorem 0.1. *Any L lines in space determine $\leq 10L^{3/2}$ joints.*

Main Lemma. *If a set of lines has J joints, then one of the lines contains $\leq 3J^{1/3}$ joints.*

The main lemma implies the theorem by removing the lines one at a time. We start with L lines and J joints. By cutting out one line, we reduce the number of joints by $\leq 3J^{1/3}$. We look at the remaining lines $L-1$ lines, which contain $\leq J$ joints. One of the lines has $\leq 3J^{1/3}$ joints on it. Removing this line, we reduce the number of joints by $\leq 3J^{1/3}$. We remove all L lines, one at a time. Each time the number of joints decreases by $\leq 3J^{1/3}$, and we end up with no joints. Therefore, $J \leq L(3J^{1/3})$. Rearranging we get $J^{2/3} \leq 3L$, which implies the theorem.

Now we turn to the proof of the main lemma.

Proof. Let P be a lowest degree non-zero polynomial that vanishes at every joint. The degree of P is $\leq 3J^{1/3}$ by dimension counting, as in Lecture 1. If every line has $> 3J^{1/3}$ joints, then P must vanish on every line.

Lemma 0.2. *If x is a joint lying in three (non-coplanar) lines, and if a smooth function $F : \mathbb{R}^3 \to \mathbb{R}$ vanishes on the lines, then ∇F vanishes at x.*

Proof. Let v_1, v_2, v_3 be tangent vectors for the three lines. The directional derivative of F in the direction v_i must vanish at x. So we have $\nabla F(x) \cdot v_i = 0$ for each i. Since the v_i are a basis of $\mathbb{R}^3$, we have $\nabla F(x) = 0$.   □

So we see that the derivates of P vanish at each joint. The derivates have smaller degree than P. Since P was a minimal degree non-zero polynomial that vanishes at each joint, the derivatives of P are all the zero polynomial! Then P must be constant, and we get a contradiction.   □

For example, suppose that we start with an $A \times B \times C$ grid of lines with $A < B < C$. The number of lines is $AB + AC + BC$, and the number of joints is ABC. All of the joints are contained in a union of A parallel planes. Therefore, there is a polynomial of degree A which vanishes on all the joints (the polynomial is a product of linear factors). This polynomial actually has minimal degree: it's an exercise to check that a polynomial of degree $\leq A - 1$ which vanishes on all the joints of this configuration must be the zero polynomial. The minimal polynomial vanishes on all the lines with B joints and all the lines with C joints, but on none of the lines with A joints. So we see that the minimal polynomial identifies the important and less important lines, and locates at least one inessential line with not too many joints on it.