THE KAKEYA PROBLEM

1 Introduction

Consider cylinders $T_i \subseteq \mathbb{R}^n$ of length N and radius 1. $\{T_i\}$ is a *Kakeya set of tubes* if $\{v(T_i)\}$ are $\frac{1}{N}$-separated and $\frac{2}{N}$-dense in S^{n-1}, where $v(T_i)$ is a unit vector "in the direction of the tube T_i". Note that this implies that the number of tubes should be $\sim N^{n-1}$.
The main question that concerns us is: How small can $|\cup T_i|$ be?
In the previous lecture we have seen Besicovitch's construction, in which $|\cup T_i| \sim \frac{1}{logN} \cdot N^n$ (We have seen in it two dimensions, but it can be done in any dimension). Is this the right order of magnitude? Can we get a polynomial compression of the tubes' volume?

Conjecture 1 *(Kakeya) For any $\epsilon > 0$, any Kakeya set of tubes $\{T_i\} \subseteq \mathbb{R}^n$ satisfies $|\cup T_i| \geqslant c_\epsilon N^{n-\epsilon}$ (that is, polynomial compression is impossible)*

Recall the finite field version of the Kakeya problem:

Theorem 2 *A Kakeya set $K \subseteq \mathbb{F}_q^n$ has at least $c_n q^n$ elements (where in $\mathbb{F}_q^n$, a set is Kakeya if it contains a line in every direction).*

The finite field version is analogous to the Kakeya conjecture if you think of $q \sim N$, which makes sense because lines in $\mathbb{F}_q^n$ have length q, and a set that contains a line in each direction should have $\sim q^{n-1}$ lines. In fact it is even more closely related to the *Segment Version* of the Kakeya problem, in which $K \subseteq \mathbb{R}^n$ is a set that contains a unit line segment in each direction (and the volume is replaced by Hausdorff dimension).
Besicovitch showed that there exist $\{K_i\}$ with $m(K_i) \to 0$ (where m denotes measure), or there exists K with $m(K) = 0$ (His construction implies the former).
In terms of Hausdorff-dimension, the Kakeya conjecture is:

Conjecture 3 *(Hausdorff-dimension Kakeya) For any $\epsilon > 0$, H-dim$(K) \geq n - \epsilon$.*

The Hausdorff-dimension conjecture implies the tube version conjecture. From now on we will discuss the tube version because it is more natural and less involved.

2 What Is Known?

- The Kakeya conjecture is true in two dimensions.
  The flavour of the argument is as follows: For any two tubes corresponding to angles θ_1, θ_2, the volume of their overlap area is $\lesssim \frac{1}{|\theta_1 - \theta_2|}$. Hence

$$\int |\sum_i \chi_{T_i}|^2 = \sum_i \sum_j \int \chi_{T_i} \cdot \chi_{T_j} = \sum_i \sum_j |T_i \cap T_j| \lesssim (logN)^2 N^2 \tag{1}$$

  Now by the Cauchy-Schwartz inequality we have

$$N^2 = \int (\sum_i \chi_{T_i}) \cdot 1 \leq (\int |\sum_i \chi_{T_i}|^2)^{\frac{1}{2}} \cdot |\cup T_i|^{\frac{1}{2}} \lesssim logN \cdot N \cdot |\cup T_i|^{\frac{1}{2}} \tag{2}$$

  Rearranging, we see that $|\cup T_i| \gtrsim N^2 (logN)^{-some\ power}$, as desired.

Conjecture 4 *(L^p version)*

$$\int |\sum_i \chi_{T_i}|^p \lesssim N^\epsilon \cdot (\textit{what happens when all tubes are centered at } 0) \tag{3}$$

This implies the volume version of the Kakeya problem by the same argument as above.

- Using the Bush/Hairbrush Arguments.
  In the finite field setting, the bush argument shows that $|K| \gtrsim q^{\frac{n+1}{2}}$, while in $\mathbb{R}^n$ it shows that $|\cup T_i| \gtrsim N^{\frac{n+1}{2}}$.
  In the finite field setting, the hairbrush argument shows that $|K| \gtrsim q^{\frac{n+2}{2}}$, while in $\mathbb{R}^n$ it shows that $|\cup T_i| \gtrsim N^{\frac{n+2}{2}}$.

  More concretely: when applying the bush argument in $\mathbb{R}^n$, tubes with a small angle between them may overlap a lot. However, if we look at a distance say $> \frac{N}{10}$ from the intersection point of all the tubes in the bush, then the tubes don't overlap a lot any more and we can use the bush argument to get $|bush| \gtrsim N \cdot (\#\textit{tubes in the bush})$.
  In the hairbrush argument, we consider each plane separately. As in the bush argument, we need to look at a distance $> \frac{N}{10}$ from the central axis of the hairbrush. Only here there might be a problem with tubes that are at a small angle from the central axis tube. In the mid 90's, Wolff was able to solve that problem and prove that $|\cup T_i| \gtrsim N^{\frac{n+2}{2}}$.

- In 3 dimensions, the hairbrush argument gives $|\cup T_i| \gtrsim N^{\frac{5}{2}}$. The best known result today is $|\cup T_i| \gtrsim N^{\frac{5}{2}+10^{-10}}$ (this is a result of Katz-Laba-Tao from ~2000, with minor assumptions about the Kakeya set).

- Wolff came up with some toy problems:
  - The finite field Kakeya problem
  - Lines in different directions in the plane (pretty easy), which led him to consider circles of different radii in the plane.

  He used incidence geometry methods such as the cellular method and ideas related to the Szemeredy-Trotter theorem.

3 Applying the Polynomial Method to the Kakeya Problem

Recall the main ideas in the proof of the finite field version: Let $K \subseteq \mathbb{F}_q^n$ be a Kakeya set, and suppose $|K| = q^{n-\gamma}$. Then

1. Find a non-zero polynomial P that vanishes on K with $deg(P) \lesssim q^{1-\frac{\gamma}{n}}$.
2. The structure of K and the choice of P imply that P vanishes at some other places.
3. Hence the non-zero polynomial P vanishes too much $\Rightarrow$ a contradiction.

Can we do something analogous to prove the Kakeya conjecture in $\mathbb{R}^n$?
Let $K \subseteq \mathbb{R}^n$ be a Kakeya set of $1 \times N$ tubes $\{T_i\}$, and suppose $|K| = N^{n-\gamma}$. Let l_i be the core line of T_i (note that these lines may be disjoint even if the tubes overlap). The following are some ideas from the class discussion regarding the choice of the polynomial P:

1. Pick P that vanishes on $\{l_i\}$.

2. Pick P that vanishes on $\{\partial T_i\}$. This will imply that P vanishes on the boundary of the infinite continuation of the tubes, and thus on an infinite surface (in fact, we know what P will have to be - the product of polynomials vanishing on the boundaries of the tubes). So the degree of P will probably be very large, and this will not serve our purpose.

3. Pick P such that $Z(P)$ (roughly) bisects each tube.

4. Pick P such that $Z(P)$ intersects or bisects each cross-section of the tubes. This could be nice (for a start, it is well defined), but it consists of infinitely many conditions on P, so it might be hard to satisfy.

5. Pick P such that P is small on K (not necessarily 0), but with some normalization such as $sup(|coef(P)|) = 1$.

6. (Following idea 4) Consider $1 \times 1 \times ... \times 1$ cubes in $\cup T_i$ and have $Z(P)$ bisect each one of them.

7. Even better - have $Z(P)$ bisect *lattice* cubes that are completely contained in $\cup T_i$. We can consider lattice cubes of size $\sim 2^{-n}$, so that each tube will contain a lot of them. In that case, the number of cubes we want to bisect is $\sim N^{n-\gamma}$ (which is the volume of K). We can use the Polynomial Ham Sandwich Theorem to find such a polynomial P with $deg(P) \lesssim N^{1-\frac{\gamma}{n}}$, so this could be an analogue for step 1 of the finite field proof. However, it is unclear whether we can have an analogue of step 2 here - can we use the fact that $Z(P)$ bisects each cube in the tubes to say anything about the behaviour of P in another cube?
   We could pick a line l in a tube. Note that l could be disjoint from $Z(P)$, but optimistically we could have $|l \cap Z(P)| = cN$, so that P must vanish on l. To make this happen, perhaps we could choose l randomly among parallel lines in the tube. In the next lecture, we will try to develop this idea.