

THE KAKEYA PROBLEM

In this lecture, we will discuss Kakeya Conjecture and some known results about it.

Definition Suppose that $T_i \subset \mathbb{R}^n$ are tubes of length N and radius 1. $\{T_i\}$ is a Kakeya set of tubes if $\{V(T_i)\}$ is $\frac{1}{N}$ -separated and $\frac{2}{N}$ -dense in S^{n-1} , where $V(T_i)$ is the unit vector of direction of the tube T_i .

Our question is: how small can $|\cup T_i|$ be?

Recall that last time we gave the "Besicovitch arrangement of tubes" where we managed to compress the volume of $\cup T_i$ by a factor of $\log N$. We got that arrangement by translating the tubes in a certain way, without performing any rotation. We want to know whether there is a better compression.

Kakeya Conjecture(tube version). For any Kakeya set of tubes $T_i \subset \mathbb{R}^n$, $|\cup T_i| \geq C_\epsilon \cdot N^{n-\epsilon}$ ($\forall \epsilon > 0$).

We also have:

Kakeya Conjecture(segment version). For any Kakeya set K (a set of points that contains a unit line segment in every direction) in \mathbb{R}^n , $\text{H-dim}(K) \geq n - \epsilon$, where $\text{H-dim}(K)$ is the Hausdorff dimension of K and ϵ is any positive number.

Notice that the segment version will imply the tube version. The tube version has a combinatorial flavor since it involves how tubes can overlap each other.

1. THE 2D CASE

Proposition. The Kakeya Conjecture(tube version) is true in dimension two.

Now we sketch the proof here. The flavor is similar to the finite field Kakeya problem.

We denote by θ_i the angle between $V(T_i)$ and the x -axis. Then the overlapping area of T_1 and T_2 can be bounded by $\frac{1}{|\theta_1 - \theta_2|}$. Then we get

$$\int |\sum \chi_{T_i}|^2 = \sum \sum \int \chi_{T_i} \chi_{T_j} = \sum \sum |T_i \cap T_j| \lesssim |\log N| N^2$$

By Cauchy-Schwarz inequality, we have $N^2 = \int (\sum \chi_{T_i}) \leq (\int |\sum \chi_{T_i}|^2)^{\frac{1}{2}} \cdot |\cup T_i|^{\frac{1}{2}}$. Thus $|\cup T_i| \gtrsim N^2 (\log N)^{-1}$.

Conjecture(L^p version). $\int |\sum \chi_{T_i}|^p \lesssim N^\epsilon$ (what happens if all tubes are centered at zero)
This will imply the Kakeya Conjecture(tube version) by a similar argument.

2. BUSH ARGUMENT AND HAIR BRUSH ARGUMENT

- Bush argument: We have $|K| \gtrsim q^{\frac{n+1}{2}}$ for a Kakeya set $K \subset \mathbb{F}_q^n$ and $|\cup T_i| \gtrsim N^{\frac{n+1}{2}}$ for a Kakeya set of tubes $T_i \subset \mathbb{R}^n$

We have already seen how Bush argument works in the finite field case. For the tube version the similar argument works too.

Suppose $|\cup T_i|$ is small, then there must be a point that is covered by many tubes. Those tubes might have a large overlapping area around that point, but if we consider what happens in a distance of $\frac{N}{10}$ from that point, then we see the volume of the bush is bigger than $N \cdot$ (the number of tubes in the bush)

- Hair Brush argument: We have $|K| \gtrsim q^{\frac{n+2}{2}}$ for a Kakeya set $K \subset \mathbb{F}_q^n$ and $|\cup T_i| \gtrsim N^{\frac{n+2}{2}}$ for a Kakeya set of tubes $T_i \subset \mathbb{R}^n$

In the finite field case, this argument goes like to choose the line that has the biggest number of intersection with other lines and consider all lines that intersect it. This will give us the bound. However it is much trickier to get the bound for the tube version: when we consider what happens in a distance of $\frac{N}{10}$ from our chosen tube, it turns out that tubes that have small angle to the chosen tube might not even make it out that distance. It is possible, though not easy, to rule out such cases and get the desired bound as was shown by Thomas Wolff in the 90s.

In 3D, the Hair Brush argument gives $|\cup T_i| \gtrsim N^{\frac{5}{2}}$. It is surprisingly hard to improve this bound. Katz-Laba-Tao, under a minor assumption about K , improved the bound to something like $N^{\frac{5}{2}+10^{-10}}$. Being stuck at this point, Thomas Wolff proposed some toy problems:

- Finite field Kakeya problem. People think that passing from \mathbb{R}^n to \mathbb{F}_q^n might make the problem a little bit easier while still preserving some of the flavor, as is shown in the hair brush argument.
- Instead of considering tubes in different directions, we can take annuli with thickness $\frac{1}{N}$ and radii between 1 and 2. In order to solve that, he used incidence geometry, stuff related to Szemerédi-Trotter theorem. That was cool because it brought a whole different set of techniques to this area, so people in harmonic analysis learned about this area of mathematics.

3. POLYNOMIAL METHOD FOR TUBE VERSION KAKEYA PROBLEM

Since we have already seen the elegant proof of finite field Kakeya problem using polynomial method, can we say anything about the tubes by using polynomial method?

Let us recall the main ideas we used when we solved the finite field Kakeya problem:

- (1) Look at the polynomial P that vanishes on K with smallest degree. The degree would be significantly smaller than q .
- (2) P must vanish at some other places, then we have contradiction.

Now let us see what happens for tubes. Suppose K is a Kakeya set of $1 \times N$ tubes $T_i \subset \mathbb{R}^n$, $|K| = N^{n-\gamma}$. Here are some ideas:

- Look at the polynomial P that vanishes at all core lines with smallest degree. But those lines can be all disjoint. Even if they are not, we can make a small perturbation to make them so.
- Look at the polynomial P that vanishes on ∂T_i . Then $P = 0$ on the infinite surfaces. But the degree of P would be very big.
- Instead of vanishing, P is just small on $\cup T_i$, with some normalization.
- $Z(P)$ bisects each tube. If it does it by cutting tubes at their mid-points, then there is not much information. We would like it to cut tubes along their core lines, but it seems that by requiring so we are putting infinitely many conditions on our polynomial.
- $Z(P)$ bisects each lattice cubes with size $\frac{1}{100}$ that overlaps our tubes. The polynomial ham sandwich theorem allows us to find one with degree $\lesssim N^{1-\frac{\gamma}{n}}$. Now our question is: does such a polynomial necessarily bisect some other cubes?

We will see what we should do next in the last class on Wednesday.