18.S997 Notes

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We defined $\tilde{K}_\alpha(x) = (1+|x|)^{-\alpha}\cos|x|$ and $\tilde{T}_\alpha f := f * \tilde{K}_\alpha$. We study a tube T with length $L \geq 1$ and diameter $L^{1/2}/1000$. Letting v_T be the unit vector along the tube, we defined $f_T(x) = e^{iv_T\cdot x}\chi_T$. We also defined T^+ to be T shifted by twice its length in the direction of v_T; that is, $T^+ = T + 2Lv_T$. We proved:

Proposition. *$\left|\tilde{T}_\alpha f_T(x)\right| \gtrsim L^{\frac{n+1}{2}-\alpha}$ on T^+.*

Corollary. *If $\alpha < \frac{n+1}{2}$, then $\left\|\tilde{T}_\alpha f\right\|_p \lesssim \|f\|_p$ is false for all p.*

We will now use these tubes to show a similar result for $\alpha = \frac{n+1}{2}$.

Theorem (Fefferman 71)**.** *$\left\|\tilde{T}_{\frac{n+1}{2}} f\right\|_p \lesssim \|f\|_p$ is false for all $p \neq 2$.*

When $p = 2$, one can prove using Fourier analysis that $\left\|\tilde{T}_{\frac{n+1}{2}} f\right\|_2 \lesssim \|f\|_2$.

Proof. We will prove Fefferman's result for $p > 2$. The idea is to look at many tubes T_i, and take $f = \sum_i f_{T_i}$. We wish to arrange disjoint tubes in such a way that their translates have large intersection. The idea for the case $p < 2$ is to start with the intersecting tubes and translate them so they are disjoint. An arrangement due to Besicovitch will be particularly useful:

Theorem (Besicovitch, 20s)**.** *For any $L \geq 1$, there exists a collection of tubes T_i (as above) such that the T_i are disjoint, but $\left|\bigcup T_i^+\right| \leq \frac{1}{K}\left|\bigcup T_i\right|$, and $\sum \chi_{T_i^+} \sim K$ on a set of size $\sim \frac{1}{K}\left|\bigcup T_i\right|$, where $K \gtrsim \frac{\log L}{\log\log L}$.*

We will prove Besicovitch's result later, but first we will use it to prove Fefferman's result.

Suppose x lies in K tubes T_i^+. Then $\tilde{T}_{\frac{n+1}{2}} f(x)$ has about K contributions of size around 1. However, not all of these are necessarily positive; to get around this, we will randomly vary the sign of our function.

Proposition. *If g_i are functions, take $g_{ran} = \sum \pm g_i$ where the signs are taken randomly. Then for all $1 \leq p \leq \infty$, $\|g_{ran}\|_p \sim \left\|\left(\sum |g_i|^2\right)^{1/2}\right\|_p$ with high probability.*

We won't prove this; the proof is similar to when we applied a Chernoff bound.

Take $f_{ran} = \sum \pm f_{T_i}$. $|f_{ran}| = 1$ on $\bigcup T_i$ since the T_i are disjoint. But $\tilde{T}_{\frac{n+1}{2}} f_{ran} \sim K^{1/2}$ on a set of size $\sim \frac{1}{K}\left|\bigcup T_i\right|$. So taking the L_p norms, $\int |f_{ran}|^p = \left|\bigcup T_i\right|$ and $\int \left|\tilde{T}_{\frac{n+1}{2}} f_{ran}\right|^p \sim K^{\frac{p}{2}} K^{-1}\left|\bigcup T_i\right|$. If $p > 2$, $\frac{\left\|\tilde{T}_{\frac{n+1}{2}} f_{ran}\right\|_p}{\|f_{ran}\|_p}$ tends to infinity with K and therefore with L, as desired.

We will now prove Besicovitch's result in the two-dimensional case. The higher dimensional cases are similar. We will rescale so that $L = 1$ and the diameter of the cylinders is N^{-1} (where N is an integer to be determined). We will define lines that will correspond to the centers of the tubes: $\ell_j(x) = \frac{j}{N}x + H(j)$

(here ℓ_j is a function form $\mathbb{R}$ to $\mathbb{R}$, and so defines a line in the plane). Let R_j be the $1/N$-neighborhood of $[\ell_j(0), \ell_j(1)]$, for $j = 0, \ldots, n-1$. These will be our tubes.

Take, for some integer A, $N = A^A$. We will work base A, so we may write $\frac{j}{N} = \sum_{a=1}^{A} j(A)A^{-a}$. Then define the heights $H(j) = -\sum_{a=1}^{A} \frac{a}{A} j(a)A^{-a}$. We will complete the proof and then present the geometric intuition. We will show that $|\bigcup R_j| \leq \frac{10}{A}$. Then $A \log A = \log N$, so $A \gtrsim \frac{\log N}{\log \log N}$. Since $N \sim \sqrt{L}$ so $A \gtrsim \frac{\log L}{\log \log L}$.

Lemma. *If $j(A) = J(a)$ for $a = 1, \ldots, b-1$, then $\left|\ell_j(\frac{b}{A}) - \ell_J(\frac{b}{A})\right| \leq 2A^{-b}$.*

Proof. We defined $H(j) = -\sum_{a=1}^{A} \frac{a}{A} j(a)A^{-a}$. Therefore, $\ell_j(x) = \sum_{a=1}^{A} \left(x - \frac{a}{A}\right) j(a)A^{-a}$. When we subtract $\ell_j(x)$ and $\ell_J(x)$, the first $b-1$ terms cancel, as for those $j(a) = J(a)$. Note the $a = b$ terms are 0 since $x = \frac{b}{A}$.

The remaining terms are bounded by:

$$\sum_{a=b+1}^{A} \left|\frac{b-a}{A} A^{-a} j(a)\right| + \sum_{a=b+1}^{A} \left|\frac{b-a}{A} A^{-a} J(a)\right| \leq \sum_{a=b+1}^{A} A^{-a} j(a) + \sum_{a=b+1}^{A} A^{-a} J(a) \leq 2A^{-b}.$$

□

Corollary. *If $j(a) = J(a)$ for $a = 1, \ldots, b-1$, $|\ell_j(x) - \ell_J(x)| \leq 4A^{-b}$ for $x \in [\frac{b-1}{A}, \frac{b}{A}]$.*

Proof. They are within $2A^{-b}$ at b/A, and their slope difference is at most $A^{-(b-1)}$ over the interval of length A^{-1}, so there is a change of at most A^{-b}, giving the (slightly stronger than) desired bound. □

Corollary. *$\bigcup R_j \cap ([\frac{b-1}{A}, \frac{b}{A}] \times \mathbb{R})$ is covered by $A^{(b-1)}$ horizontal strips (really, parallelograms) of width $6A^{-b}$.*

Proof. There are A^{b-1} choices for $j(1), \ldots, j(b-1)$. Then there is a strip within these of this width that covers all of the possible lines. □

Corollary. *The area of the R_j is at most $10A^{-1}$.*

Proof. Just add up the area within each strip according to the last corollary. □

To check that the R_j give the desired construction, we still need that their translates are disjoint and that most points are contained in many of them. We leave this as an exercise to the reader. □

We are also interested in $\tilde{T}_\alpha$ for $\alpha > \frac{n+1}{2}$. However, this remains open.

Conjecture (Bochner-Riesz)**.** *If $\alpha > \frac{n+1}{2}$, then $\left\|\tilde{T}_\alpha f\right\|_p \lesssim \|f\|_p$ for $\frac{n}{\alpha} < p < \frac{n}{n-\alpha}$.*

One could try to apply an argument similar to Fefferman's to contradict this. In particular, Fefferman's argument shows that the Bochner-Riesz conjecture implies:

Conjecture. *If T_i are tubes of length L as above, $\epsilon > 0$, the T_i disjoint, then $\left|\bigcup T_i^+\right| \geq c_\epsilon L^{-\epsilon} \left|\bigcup T_i\right|$.*