OSCILLATING INTEGRALS AND THE KAKEYA PROBLEM

1. The ball multiplier in Fourier analysis

We give a little background in Fourier analysis.

The Fourier transform in $\mathbb{R}^n$ is defined by

$$\hat{f}(\omega) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \omega x} dx.$$

In this short background section, we will assume that f is continuous and compactly supported. With these assumptions, the integral above is clearly defined. A function can be recovered from its Fourier transform as follows:

Proposition 1.1. *If f is a smooth compactly supported function, then*

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\omega) e^{2\pi i \omega x} d\omega.$$

As long as f is C^∞ smooth, $\hat{f}$ decays rapidly, and this integral is defined. If f is just continuous with compact support, then $\hat{f}$ is a continuous function, but it may not be integrable. In this case, it requires thought to understand what the right-hand side should mean. Partly for this reason, Fourier analysts considered integrating just over a ball:

$$M_R f(x) := \int_{B^n(R)} \hat{f}(\omega) e^{2\pi i \omega x} d\omega.$$

If f is continuous with compact support, then $M_R f$ is well defined for any finite R. It is natural to ask whether $M_R f$ converges to f as $R \to \infty$. Here are some fundamental results about this question.

- At a particular point x, $M_R f(x)$ may not converge at all. (19th century)
- The functions $M_R f$ converge to f in L^2 (in every dimension). (One of the motivations for defining L^2 convergence in the early 20th century)
- If $n = 1$, then $M_R f$ converges to f in L^p for every $1 < p < \infty$. (Riesz, early 20th century.)

Question: For a given dimension n, for which p do we have $M_R f \to f$ in L^p for all $f \in C^0_{comp}$?

The operators M_R are all in a family, and if one gets a good understand of M_1, then by rescaling one can also get a good understanding of any M_R. By standard analysis tricks, this question is equivalent to the following:

Question: For a given dimension n, for which p do we have $\|M_1 f\|_p \lesssim \|f\|_p$?

2. Oscillating kernels

Last time we considered the kernel $K_\alpha(x) := |x|^{-\alpha}$. Now we consider an oscillating version of this kernel.

$$\tilde{K}_\alpha(x) := [1 + |x|]^{-\alpha} \cos |x|.$$

The function $\tilde{K}_\alpha(x)$ is still radial. Near the origin, it's bounded instead of having a sharp peak. Also, it oscillates with the radius, so that it has positive and negative parts. If one dropped a stone into a pond and looked at the ripples, the shape would be a little bit like $\tilde{K}_\alpha$, with a modest peak in the center, and then waves going outward and getting smaller the farther they are from the center.

We define $\tilde{T}_\alpha f := f * \tilde{K}_\alpha$.

The operator M_1 turns out to be very similar to $\tilde{T}_{\frac{n+1}{2}}$. Although they are not exactly equal, all the arguments that we will make about $\tilde{T}_{\frac{n+1}{2}}$ apply just as well to M_1. From now on, we'll just talk about $\tilde{T}_\alpha$.

Our main question is the following, what are all the L^p estimates obeyed by $\tilde{T}_\alpha$?

At first sight, this problem looks like a small variation on the Hardy-Littlewood-Sobolev problem - it's just a similar kernel with some oscillations added. Because of the oscillations, there are positive and negative terms in the integrals, and some cancellation occurs. The key issue is to understand how much cancellation needs to occur.

We will focus on estimates of the form $\|T_\alpha f\|_p \lesssim \|f\|_p$, so that we have less parameters to keep track of. ($L^p - L^q$ estimates are interesting too, but all of the essential issues already appear in this main case.)

Example 1. We let $f = \chi_{B_r}$ for some r. It's already somewhat complicated to estimate $\tilde{T}_\alpha f$ because of the cancellation in the integral. But if f is $< 1/100$, then at most points x, there is no cancellation in the integral

$$\tilde{T}_\alpha f(x) = \int_{B(r)} [1 + |x - y|]^{-\alpha} \cos |x - y| dy.$$

The most interesting is to take $r = 1/100$. In this case, $\tilde{T}_\alpha f \sim \tilde{K}_\alpha$. In this case, we have $\|f\|_p \sim 1$, and $\int |\tilde{T}_\alpha f|^p \sim \int_{\mathbb{R}^n} (1 + |x|)^{-\alpha p}$. So $\|\tilde{T}_\alpha f\|_p < \infty$ iff $\alpha p > n$.

Considering $r \leq 1/100$ just gives the same information.

Example 2. (Focusing example) For large r, there is something better to do than χ_{B_r}. Suppose that we want to make $\tilde{T}_\alpha f(0)$ large. Let's write it out as an integral:

$$\tilde{T}_\alpha f(0) = \int_{\mathbb{R}^n} f(y)[1+|y|]^{-\alpha} cos|y| dy.$$

If we choose f carefully, then all the contributions in the integral are positive, instead of cancelling each other. This motivates choosing $f_2 = \chi_{B_r} sign(cos|y|)$, for some large $r \geq 1$. We have $\|f_2\|_p = r^{n/p}$. We also have $|\tilde{T}_\alpha f_2(0)| \sim r^{n-\alpha}$. In fact, for all $|x| < 1/100$, we have $|\tilde{T}_\alpha f_2(x)| \sim r^{n-\alpha}$. Therefore, $\|\tilde{T}_\alpha f_2\|_p \gtrsim r^{n-\alpha}$. So $\|\tilde{T}_\alpha f_2\|_p \lesssim \|f_2\|_p$ iff $n/p \geq n - \alpha$.

In summary, we have the following proposition.

Proposition 2.1. *If $\|\tilde{T}_\alpha f\|_p \lesssim \|f\|_p$ for all the examples above, then*

$$\frac{n}{\alpha} < p \leq \frac{n}{n-\alpha}.$$

Exercise. Being a little more clever/careful in Example 2., we can get eliminate the upper endpoint. If $\|\tilde{T}_\alpha f\|_p \lesssim \|f\|_p$ for all f, then

$$\frac{n}{\alpha} < p < \frac{n}{n-\alpha}.$$

(If $n/p = n - \alpha$, we can consider $f_3 = \chi_{B_r} \tilde{K}_{n-\alpha}$. This rules out the endpoint, leaving only $n/p > n - \alpha$.)

If particular, if $\alpha = (n+1)/2$, then we have a bound on all examples provided that $\frac{2n}{n+1} < p < \frac{2n}{n-1}$. This was the situation until the early 70's.

3. Examples shaped like tubes

There is another important example in the theory of these operators: an oscillating function supported on a long thin tube.

Let T be a cylinder of length $L >> 1$ and radius $(1/1000)L^{1/2}$. The cylinder may point in any direction. Let v_T be a unit vector parallel to the axis of the cylinder. Let f_T be the function

$$f_T(x) := \chi_T(x) e^{i(v_T \cdot x)}.$$

We want to understand $\tilde{T}_\alpha f_T$. Let T^+ denote the cylinder we get by translating T by $2Lv_T$. The most interesting part is the behavior of $\tilde{T}_\alpha f_T$ on T^+. Consider a point x in T^+.

$$\tilde{T}_\alpha f_T(x) = \int_T |x-y|^\alpha \cos|x-y| e^{i(v_t \cdot y)} dy.$$

Now the key point is that the oscillations of $e^{i(v_t \cdot y)}$ and the oscillations of $\cos|x-y|$ are in sync on T. Let's consider the set where $e^{iv_t \cdot y}$ is equal to 1 – this set is the set of peaks of the real part of the wave $e^{iv_t \cdot y}$. We have $e^{iv_t \cdot y} = 1$ when $v_t \cdot y = 2\pi n$, $n \in \mathbb{Z}$. This set is a union of parallel planes, perpendicular to the axis of T with spacing 2π between them. The peaks of the wave $cos|x-y|$ occur at $|x-y| = 2\pi n$, on spheres around x with radius $2\pi n$. But inside of the tube T, each sphere looks almost like a plane. It's a good idea at this point to draw a picture of the level sets of $v_t \cdot y$ and of $|x-y|$ inside of T. Because of this, the two waves interfere constructively. Let's examine the situation more computationally now.

The vector $x - y$ is nearly parallel to v_t. The v_t component of $x-y$ is $\sim L$, and the perpendicular component is $\leq (1/1000)L^{1/2}$. By the Pythagorean theorem, we have

$$(v_t \cdot x - v_t \cdot y)^2 - 10^{-4} L \leq |x-y|^2 \leq (v_t \cdot x - v_t \cdot y)^2 + 10^{-6} L$$

Since $|v_t \cdot x - v_t \cdot y| \geq L/4$, we see that

$$|x-y| - |v_t \cdot x - v_t \cdot y| \leq 10^{-5}.$$

Therefore, up to a small error, we have

$$\tilde{T}_\alpha f_T(x) = \int_T |x-y|^\alpha \cos(v_t \cdot x - v_t \cdot y) e^{i(v_t \cdot y)} dy + \text{ small error.}$$

Expanding $\cos a = (1/2)(e^{ia} + e^{-ia})$, we get

$$\tilde{T}_\alpha f_T(x) = (1/2) e^{iv_t \cdot x} \int_T |x-y|^{-\alpha} dy + (1/2) e^{-iv_t \cdot x} \int_T |x-y|^{-\alpha} e^{2iv_t \cdot y} dy + \text{ small error.}$$

The first integral is the main term. There's lots of cancellation in the second integral, so it's much smaller. The error term is bounded by $\int_T |x-y|^{-\alpha} 10^{-5} dy$, so it's much smaller than the main term.

The volume of T is $\sim L^{\frac{n+1}{2}}$, and $|x-y| \sim L$, so the main term has size $\sim L^{\frac{n+1}{2} - \alpha}$.

Proposition 3.1. *If f_T and T^+ are defined as above, then for every $x \in T^+$ we have*

$$|\tilde{T}_\alpha f_T(x)| \gtrsim L^{\frac{n+1}{2} - \alpha}.$$

Corollary 3.2. *If $\alpha < \frac{n+1}{2}$, then there are no bounds of the form $\|\tilde{T}_\alpha f\|_p \lesssim \|f\|_p$.*

Proof. Notice that T^+ has the same size as T. The function f_T has size ~ 1 and support on T. If $\alpha < \frac{n+1}{2}$, then the function $\tilde{T}_\alpha f_T$ has size $>> 1$ on T^+. So $\|\tilde{T}_\alpha f_T\|_p \sim L^{\frac{n+1}{2} - \alpha} \|f_T\|_p$. □

This type of example appears in a number of linear operators besides $\tilde{T}_\alpha$. There are similar examples connected to the wave equation. It takes some work to write them down precisely, but we can give some feel for it just in words. Imagine an airplane traveling at the speed of sound. The path of the airplane in space-time is like a long thin tube. The engine of the plane vibrates, making sound waves, and these sound waves travel at the same speed as the airplane. The airplane can feel dramatically stronger sound waves than it would have felt at a lower or higher speed. Even if the airplane turns off the engine, there will still be strong sound waves in the plane for some time. The action of the engine occurs on one tube in space time, and the resulting sound waves have large amplitude on a longer tube. Although the operator $\tilde{T}_\alpha$ is not an accurate model for sound waves, the mathematical issues in understanding it are similar with those in the wave equation.

We now return to our operators $\tilde{T}_\alpha$. For $\alpha \geq (n+1)/2$, we have $\|\tilde{T}_\alpha f_T\|_p \lesssim \|f_T\|_p$ for all p. In particular, the ball multiplier M_1 is similar to $\tilde{T}_{(n+1)/2}$, and we have $\|M_1 f_T\|_p \sim \|f_T\|_p$ for all p as well. So this example does not give any new information about the ball multiplier. For all the examples we have considered so far, we have

$$\|M_1 f\|_p \lesssim \|f\|_p, \text{ for all } \frac{2n}{n+1} < p < \frac{2n}{n-1}.$$

Until the early 70's, it was generally believed that these inequalities were true. The only case that was proven was $p = 2$. In "The multiplier problem for the ball", Charles Fefferman proved that these inequalities are false for all $p \neq 2$. (The paper appeared in Ann. of Math. (2) 94 (1971), 330-336). These counterexamples are given by arranging many tubes in a remarkable pattern found by Besicovitch.

4. Sums of many tubes

Let us consider a function $f = \sum_i f_{T_i}$ over many tubes T_i. Then we have $\tilde{T}_\alpha f = \sum_i \tilde{T}_\alpha f_{T_i}$. Schematic picture: draw some tubes T_i in blue, and T_i^+ in red. For example, T_i may be disjoint and T_i^+ may intersect.

In the 1920's, Besicovitch constructed an arrangement of tubes so that T_i are disjoint and T_i^+ intersect a lot.

Theorem 4.1. *(Besicovitch, 1920's) Fix a dimension $n \geq 2$. For any $L \geq 1$, there is a finite set of disjoint tubes T_i (with length L and radius $\sim (1/1000)L^{1/2}$), with the property that*

$$|\cup_i T_i^+| \lesssim (\log L)^{-1} |\cup_i T_i|.$$

We'll prove Besicovitch's theorem next class (or maybe something a touch weaker). The key point for the moment is that $(\log L)^{-1}$ can be arbitrarily small.

Let $f = \sum_i f_{T_i}$, where the T_i are the tubes in Besicovitch's construction. How big is $\tilde{T}_{(n+1)/2} f$? Suppose that x lies in A tubes T_i^+. We have a sum of A terms of size ~ 1, but these terms are complex numbers that may point in any direction. We would actually have to be quite lucky if the sum of A terms had size $\sim A$. The sum of A random numbers $|z| \le 1$ has size $\sim A^{1/2}$. So we should expect something more like

$$|\tilde{T}_{(n+1)/2} f(x)| \sim \left(\sum_i |\tilde{T}_{(n+1)/2} f_{T_i}(x)|^2 \right)^{1/2} \qquad (*).$$

In fact, if we define $f_{ran} = \sum_i \pm f_{T_i}$ with random $\pm$ signs, then (*) is true with very high probability.

Proposition 4.2. *If g_i are any functions, then with high probability,*

$$\| \sum_i \pm g_i \|_p \sim \| (\sum_i |g_i|^2)^{1/2} \|_p .$$

We defer this – the probability argument is similar to one earlier in the course.

With these tools in hand, we can understand $\|f_{ran}\|_p$ and $\|\tilde{T}_\alpha f_{ran}\|_p$.

Corollary 4.3. *If T_i is any set of tubes, and $f_{ran} := \sum_i \pm f_{T_i}$, then with high probability*

$$\|f_{ran}\|_p \sim \| (\sum_i \chi_{T_i}^2)^{1/2} \|_p \sim \| \sum_i \chi_{T_i} \|_{p/2}^{1/2} .$$

In Besicovitch's example, the tubes T_i are disjoint, and so $\|f_{ran}\|_p \sim | \cup T_i|^{1/p}$.

Corollary 4.4. *If T_i is any set of tubes of length L, and $f_{ran} = \sum_i \pm f_{T_i}$, then with high probability*

$$\|\tilde{T}_\alpha f_{ran}\|_p \gtrsim L^{\frac{n+1}{2} - \alpha} \| \sum_i \chi_{T_i^+} \|_{p/2}^{1/2} .$$

In Besicovitch's example, $\sum_i \chi_{T_i^+}$ is supported on a set of measure $\lesssim (\log L)^{-1} |\cup_i T_i|$, and so its average height is $\gtrsim \log L$. Therefore, for $q > 1$, its L^q norm is $\gtrsim (\log L)^q (\log L)^{-1} |\cup_i T_i|$, and we get

$$\|\tilde{T}_\alpha f_{ran}\|_p \gtrsim L^{\frac{n+1}{2} - \alpha} (\log L)^{\frac{p-2}{4}} |\cup_i T_i|^{1/p} .$$

We get

Theorem 4.5. *(Fefferman 1971) If $p > 2$, then $\tilde{T}_{(n+1)/2}$ is not bounded on L^p.*

Exercise. The operator $\tilde{T}_{(n+1)/2}$ is also not bounded on L^p for $p < 2$. To see this, choose T_i so that T_i^+ are disjoint and $| \cup_i T_i^+|$ is much larger than $| \cup_i T_i|$.

HLS problem: connected with how balls overlap in space

BR problem: connected with how tubes overlap in space.