THE FINITE-FIELD NIKODYM AND KAKEYA PROBLEMS

These notes are rougher than I would like, but they still have some main proofs.

Let F be a finite field with q elements. A set $N \subset F^n$ is called a (generalized) Nikodym set, if for each point $x \in F^n$, there is a line $L(x)$ containing x so that $|L(x) \cap N| \geq q/2$. A trivial example of a Nikodym set is the entire set F^n. Can one find a significantly smaller Nikodym set?

Theorem 0.1. *(Dvir) Any (generalized) Nikodym set in F^n contains at least $c_n q^n$ elements.*

Proof. Let $V(d)$ be the vector space of degree d polynomials in n variables with coefficients in F. We pick a polynomial P (not identically zero) that vanishes on N with degree $\leq C_n |N|^{1/n}$. If this degree is $< q/2$, then P must vanish at every point $x \in F^n$. But a polynomial of degree $\leq q/2$ cannot vanish at every point of F^n unless it vanishes identically. We conclude that the degree of P is at least $q/2$ and so $|N| \geq C_n^{-1}(q/2)^n$. $\square$

(We used the following lemma. A polynomial P in n variables over F of degree $d < q$ cannot vanish at every point of F^n unless each coefficient of P is zero.

proof by induction on n. The case $n = 1$ appears in Lecture 2.

Suppose P vanishes at each point of F^n. Write $P(x_1, ..., x_n) = \sum_{j=0}^{d} P_j(x_1, ..., x_{n-1}) x_n^d$. For each particular choice of $x_1, ..., x_{n-1}$, we know that $P(x_1, ..., x_n) = 0$ for all $x_n \in F$. Since $d < q$, we see that the coefficients $P_j(x_1, ..., x_{n-1})$ must vanish for each j. Therefore $P_j(x_1, ..., x_{n-1}) = 0$ for each $(x_1, ..., x_{n-1}) \in F^{n-1}$. By induction, we see that the coefficients of P_j all vanish. But then the coefficients of P all vanish.)

Dvir also proved a small variation which is a tiny bit harder than the proof above. A set $K \subset F^n$ is called a Kakeya set if it contains a line in every direction. In other words, for every vector $a \in F^n \setminus \{0\}$, there is a vector b so that the line $\{at + b | t \in F\}$ is contained in K. A trivial example of a Kakeya set is the entire vector space F^n. Can one find a Kakeya set significantly smaller than this?

Theorem 0.2. *A Kakeya set $K \subset F^n$ has at least $c_n q^n$ elements.*

Proof. Let K be a Kakeya set. If K is smaller than the conclusion, let P be a polynomial vanishing on K of degree $< q$. Let d be the degree of P. Write $P = P_d + Q$, where P_d is the sum of monomials of degree d and Q is a polynomial of degree $\leq d-1$.

Let a be any non-zero vector. Choose b so that the line $\{at + b | t \in F\}$ is contained in K. Consider the polynomial in one variable $R(t) := P(at + b)$. The polynomial

R vanishes for each $t \in F$. It has degree $\leq d < q$, and so its coefficeints all vanish. In particular, its coefficient of degree d vanishes. But the coefficient of t^d in R is exactly $P_d(a)$. So we see that $P_d(a)$ vanishes for all $a \in F^n \setminus \{0\}$. But since the degree of P_d is $d < q$, it easily follows that P_d vanishes at 0 also. Then we see that P_d is identically zero, and we reach a contradiction. $\qquad\square$

The Kakeya and Nikodym problems presented here are the analogues of deep open problems in Euclidean space. A Kakeya set $K \subset \mathbb{R}^n$ is a set which contains a unit line segment in each direction. For example, the ball of radius $1/2$ is a Kakeya set. Besicovitch constructed surprising examples of Kakeya sets with arbitrarily small volume and even with measure 0. Besicovitch's construction works in each dimension $n \geq 2$. Although his sets have measure zero, they all have full Hausdorff dimension. The Kakeya conjecture is that every Kakeya set $K \subset \mathbb{R}^n$ has Hausdorff dimension n.

It can be hard to appreicate the polynomial method proofs without some background trying to prove this type of result without it. We give two simple combinatorial estimates for the size of a finite field Kakeya set $K \subset \mathbb{F}_q^n$.

1. (Bush method) By pigeonholing, there is a point $x \in K$ which lies in at least $q^n/|K|$ lines of the Kakeya set. The union of all the lines of the Kakeya set thru a given point is called a bush. All of these lines are disjoint except at x. Therefore, the bush contains at least $q^n(q-1)/|K|$ points. Since the bush is contained in K, we see that $|K| \geq (1/2)q^{\frac{n+1}{2}}$.

2. (L^2-method, or just counting) Consider the lines of K one at a time. The first contains q points. The second must contain at least $q-1$ points not in the first. The third must contain at least $q-2$ points not in the first two, etc. Therefore, the first q lines must contain at least $(1/2)q^2$ distinct points. So $|K| \geq (1/2)q^2$.

Both these methods only use the fact that two distinct lines intersect in ≤ 1 point and that a Kakeya set is the union of $\geq q^{n-1}$ distinct lines. In other words, we have not used the fact that the lines point in different directions! However, there are q^2 lines in a plane. In order to see that a Kakeya set in $\mathbb{F}^3$ must contain $q^{2+\epsilon}$ points, we need to use that the lines point in different directions in order to rule out the example that they all lie in a plane.