1. Loomis-Whitney Inequality

Let $X$ be a set of unit cubes in the unit cubical lattice in $\mathbb{R}^n$, and let $|X|$ be its volume. Let $\Pi_j$ be the projection onto the $x_j$ hyperplane. The motivating question is: if $\Pi_j$ is small for all $j$, what can we say about $|X|$?

**Theorem 1.1** (Loomis-Whitney 50’s). If $|\Pi_j(X)| \leq A$, then $|X| \lesssim A^{n-1}$.

**Remark.** The sharp constant in the $\lesssim$ is 1. The original proof is by using Holder’s inequality repeatedly.

Define a *column* to be the set of cubes obtained by starting at any cube and taking all cubes along a line in the $x_j$-direction.

**Lemma 1.2** (Main lemma). If $\sum |\Pi_j(X)| \leq B$, then there exists a column of cubes with between 1 and $B^{n-1}$ cubes of $X$.

**Proof.** Suppose not, so every column has $> B^{n-1}$ cubes. This means that there are $> B^{\frac{1}{n-1}}$ cubes in some $x_1$-line. Taking the $x_2$-lines through those, there are $> B^{\frac{2}{n-1}}$ cubes in some $x_1, x_2$-plane, and so on. Repeating this $n-1$ times, we get $> B$ cubes in the $x_1, \ldots, x_{n-1}$-plane, a contradiction. \qed

**Corollary 1.3.** If $\sum_j |\Pi_j(X)| \leq B$, then $|X| \leq B^{\frac{n}{n-1}}$.

**Proof.** Let $X'$ be $X$ with its smallest column removed. Then $\sum |\Pi_j(X')| \leq B - 1$, so by induction we get $|X'| \leq (B - 1)^{\frac{n}{n-1}}$, hence $|X| \leq B^{\frac{1}{n-1}} + |X'|$. \qed

Note that Corollary 1.3 implies Theorem 1.1.

**Theorem 1.4** (more general Loomis-Whitney). If $U$ is an open set in $\mathbb{R}^n$ with $|\Pi_j(U)| \leq A$, then $|U| \lesssim A^{\frac{n}{n-1}}$.

**Proof.** Take $U_\varepsilon \subset U$ be a union of $\varepsilon$-cubes in $\varepsilon$-lattice. Then $|U_\varepsilon| \lesssim A^{\frac{n}{n-1}}$ and $|U_\varepsilon| \to |U|$. \qed

**Corollary 1.5** (Isoperimetric inequality). If $U$ is a bounded open set in $\mathbb{R}^n$, then

$$Vol_n(U) \lesssim Vol_{n-1}(\partial U)^{\frac{n}{n-1}}.$$
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Proof. By projection onto translates of each $x_j$-hyperplane, we see that $|\Pi_j(U)| \leq \text{Vol}_{n-1}(\partial U)$, so we may apply Theorem 1.4. □

Remark. The fact that $U$ was bounded was used to define the projection of $U$ onto translates of each $x_j$-hyperplane.

2. Sobolev Inequality

Let $u \in C^1_{\text{comp}}(\mathbb{R}^n)$ satisfy $\int |\nabla u| = 1$. How big can $u$ be? We would like to find the right notion of size for $u$ that answers this question.

Theorem 2.1 (Sobolev inequality). If $u \in C^1_{\text{comp}}(\mathbb{R}^n)$, then

$$||u||_{L_{\frac{n}{n-1}}} \lesssim ||\nabla u||_{L^1}. $$

Here, the $L^p$-norm $||u||_{L^p}$ is given by

$$||u||_{L^p} = \left( \int |u|^p \right)^{1/p}$$

so that $||h \cdot \chi_A||_p = h \cdot |A|^{1/p}$. For some context about $L^p$-norms, for a function $u$, let $S(h) := \{ x \in \mathbb{R}^n \mid |u(x)| > h \}$.

Proposition 2.2. If $||u||_p \leq M$, then $|S(h)| \leq M^p h^{-p}$.

Proof. Just estimate $M^p = \int |u|^p \geq h^p |S(h)|$. □

We now prove the Sobolev inequality. A first try is the following bound.

Lemma 2.3. If $u \in C^1_{\text{comp}}(\mathbb{R}^n)$, $|\Pi_j(S(h))| \leq h^{-1} \cdot ||\nabla u||_{L^1}$.

Proof. For $x \in S(h)$, take a line $\ell$ in the $x_j$-direction. It eventually reaches a point $x'$ where $u = 0$, so $\int_\ell |\nabla U| \geq h$ by the fundamental theorem of calculus. This means that

$$||\nabla u||_{L^1} \geq \int_{\Pi_j(S(h)) \times \mathbb{R}} |\nabla u| = \int_{\Pi_j(S(h))} \int_{\mathbb{R}} |\nabla u| dx_j dx_{other} \geq |\Pi_j(S(h))| \cdot h. $$

If we apply Theorem 1.4 to the output of Lemma 2.3, we see that

$$|S(h)| \lesssim h^{-\frac{n}{n-1}} \cdot ||\nabla u||_{L^1}^{\frac{n}{n-1}},$$

which looks like the output of Proposition 2.2. So we would like to establish something like the converse in this case. For this, we require a more detailed analysis.

Lemma 2.4 (Revised version of Lemma 2.3). Let $S_k := \{ x \in \mathbb{R}^n \mid 2^{k-1} \leq |u(x)| \leq 2^k \}$. If $u \in C^1_{\text{comp}}(\mathbb{R}^n)$, then we have

$$|\Pi_j S_k| \lesssim 2^{-k} \int_{S_{k-1}} |\nabla u|.$$
Proof. For $x \in S_k$, draw a line $\ell$ in the $x_j$-direction through $x$. There is a point $x'$ on $\ell$ with $u(x') = 0$. Between $x$ and $x'$, there is some region on $\ell$ where $|u|$ is between $2^{k-2}$ and $2^{k-1}$. Then we see that along each such $\ell$, we have

$$\int_{S_k \cap \ell} |\nabla u| \geq \frac{1}{4} 2^k.$$  

Summing this along all $\ell$ perpendicular to a translate of the $x_j$-hyperplane yields the result. \qed

**Corollary 2.5.** $|S_k| \lesssim 2^{-k \frac{n}{n-1}} \left( \int_{S_{k-1}} |\nabla u| \right)^{\frac{n}{n-1}}.$

**Proof.** Put Lemma 2.4 into Theorem 1.4. \qed

**Proof of Theorem 2.1.** Take the estimate

$$\int |u|^\frac{n}{n-1} \sim \sum_{k=-\infty}^{\infty} |S_k| 2^{k \frac{n}{n-1}} \lesssim \sum_k \left( \int_{S_{k-1}} |\nabla u| \right)^{\frac{n}{n-1}} \leq \left( \int_{\mathbb{R}^n} |\nabla u| \right)^{\frac{n}{n-1}},$$

where in the last step we move the sum inside the $\frac{n}{n-1}$-power. \qed

**Remark.** The sharp constant in Theorem 2.1 is provided by a smooth approximation to a step function where the width of the region of smoothing is very small.

### 3. $L^p$ Estimates for Linear Operators

If $f, g : \mathbb{R}^n \to \mathbb{R}$ or $\mathbb{C}$, define the **convolution** to be

$$(f \ast g)(x) = \int_{\mathbb{R}^n} f(y) g(x - y) dy.$$  

We can explain this definition by the following story. Suppose there is a factory at 0 which generates a cloud of pollution centered at 0 described by $g(-y)$. If the density of factories at $x$ is $f(x)$, then the final observed pollution is $f \ast g$.

We would like to study linear operators like $T_\alpha f := f \ast |x|^{-\alpha}$, which means explicitly that

$$T_\alpha f(x) = \int f(y) |x - y|^{-\alpha} dy.$$  

We will take $\alpha$ in the range $0 < \alpha < n$, so that if $f \in C^0_{\text{comp}}$ then the integral converges for each $x$. Operators like these occur frequently in PDE. Another example is the initial value problem for the wave equation.

**Example.** Let us first see how $T_\alpha$ behaves on some examples for $f$.  

1. $\chi_{B_1}$, where $B_r$ is the ball of radius $r$. We see that

$$|T_\alpha \chi_{B_1}(x)| \sim \begin{cases} 1 & |x| \leq 1 \\ |x|^{-\alpha} & |x| > 1. \end{cases}$$

2. $\chi_{B_r}$. We see that

$$|T_\alpha \chi_{B_r}(x)| \sim \begin{cases} r^n \cdot r^{-\alpha} & |x| \leq r \\ r^n \cdot |x|^{-\alpha} & |x| > r. \end{cases}$$

2.1 $\delta$, the delta function. Morally, this is given by $\lim_{n \to \infty} r^{-n} \chi_{B_r}$.

A question we would like to ask about $T_\alpha$ is the following. Fix $\alpha$ and $n$. For which $p, q$ is there an inequality

$$||T_\alpha f||_q \lesssim ||f||_p$$

for all choices of $f$?

In some sense, this measures how much bigger $T_\alpha$ can make $f$. First, we determine the answer in Examples 1 and 2. For Example 1, $||\chi_{B_1}||_p \sim 1$, and

$$||T_\alpha \chi_{B_1}||_q \sim \int_{\mathbb{R}^n} (1 + |x|)^{-\alpha q} dx,$$

which is finite if and only if $\alpha q > n$. So (1) holds in Example 1 if and only if $\alpha q > n$. Let us assume this from now on.

For Example 2, $||\chi_{B_r}||_p \sim r^{n/p}$. For $||T_\alpha \chi_{B_r}||_q$, the value is given by two terms, one coming from the ball $|x| \leq r$ and the outside tail. The condition $\alpha q > n$ says that the contribution of the tail is finite, so we get the estimate

$$||T_\alpha \chi_{B_r}||_q \sim ||r^{-\alpha} \chi_{B_r}||_q \sim r^{n-\alpha+n/q}.$$

Thus, we conclude that (1) holds in Example 2 if and only if $\alpha \cdot q > n$ and $r^{n/p} \lesssim r^{n-\alpha+n/q}$ for all $r > 0$. The latter condition is equivalent to $n/p = n - \alpha + n/q$.

For a general linear operator $T$, we would like to ask whether

$$||Tf||_q \lesssim ||f||_p$$

under the conditions that $\alpha \cdot q > n$ and $n/p = n - \alpha + n/q$. If the answer is yes, we conclude that the characteristic functions of balls are in some sense typical for the action of $T$; otherwise, we would like to understand which functions $f$ this fails for.