## ANALYSIS AND COMBINATORICS

The last unit of the course will focus on the Kakeya problem. In this lecture, we'll give some background in analysis and look at how combinatorics interacts with analysis.

### 1. The Loomis-Whitney inequality

The Loomis-Whitney inequality says that if the projections of a set $U \subset \mathbb{R}^n$ onto the $x_j^\perp$ hyperplanes are small, then $U$ itself is small. In this section we'll see why this is true from a combinatorial standpoint.

Consider the unit cubical lattice in $\mathbb{R}^n$; i.e., the set of cubes $\{Q_\alpha\}_{\alpha \in \mathbb{Z}^n}$ where $Q_\alpha = \{x \in \mathbb{R}^n : \alpha_i \le x_i \le \alpha_i + 1 \text{ for all } i\}$. Let $X \subset \{Q_\alpha\}_{\alpha \in \mathbb{Z}^n}$. Let $\pi_j$ be the projection onto the $x_j^\perp$ hyperplane.

**Theorem 1.1** (Loomis-Whitney, 1950's)**.** *If $|\pi_j X| \le A$ for all $j$, then $|X| \lesssim A^{\frac{n}{n-1}}$.*

In fact, Loomis and Whitney's proof gives $|X| \le A^{\frac{n}{n-1}}$, with equality holding for a cube. Their proof involves using Hölder's inequality a lot. We'll give a proof that doesn't achieve the sharp constant, but uses only combinatorics.

Define a *column* in the $x_j$-direction to be the set of cubes along an $x_j$-direction line in the unit cubical lattice.

**Lemma 1.2.** *If $|\pi_j X| \le B$ for all $j$, then there is a column with $\ge 1$ and $\le B^{\frac{1}{n-1}}$ cubes of $X$.*

*Proof.* Suppose every column contains 0 or $> B^{\frac{1}{n-1}}$ cubes of $X$. Let $Q_0 \in X$ and take a column containing $Q_0$ in the $x_1$-direction; this column contains $> B^{\frac{1}{n-1}}$ cubes of $X$. For each $Q_1 \in X$ in this column, take a column containing $Q_1$ in the $x_2$-direction so that all of the new columns are parallel; we get an $x_1x_2$-plane which contains $> B^{\frac{2}{n-1}}$ cubes of $X$. For each cube in this plane, take parallel columns in the $x_3$-direction, and so on. We obtain an $x_1 \ldots x_{n-1}$-plane with $> B$ cubes of $X$, a contradiction. $\square$

**Corollary 1.3.** *If $\sum_j |\pi_j X| \le B$, then $|X| \le B^{\frac{n}{n-1}}$.*

*Proof.* Induct on $B$. Obtain $X'$ by removing from $X$ a column which contains $\ge 1$ and $\le B^{\frac{1}{n-1}}$ cubes of $X$. Then $\sum_j |\pi_j X'| \le B - 1$, so by induction $|X'| \le (B-1)^{\frac{n}{n-1}}$. Then $|X| \le |X'| + B^{\frac{1}{n-1}} \le B^{\frac{n}{n-1}}$. $\square$

Corollary 1.3 implies Theorem 1.1 when $B = nA$.

**Theorem 1.4** (Loomis-Whitney)**.** *If $U$ is an open set in $\mathbb{R}^n$ with $|\pi_j U| \leq A$ for all $j$, then $|U| \lesssim A^{\frac{n}{n-1}}$.*

*Proof.* Approximate $U$ by a union $U_\epsilon \subset U$ of $\epsilon$-cubes in an $\epsilon$-lattice. By Theorem 1.1, $|U_\epsilon| \lesssim A^{\frac{n}{n-1}}$. Taking $\epsilon \to 0$ yields the result. □

**Corollary 1.5** (Isoperimetric inequality)**.** *If $U$ is a bounded open set in $\mathbb{R}^n$, then $\mathrm{Vol}_n\, U \lesssim (\mathrm{Vol}_{n-1}\, \partial U)^{\frac{n}{n-1}}$.*

*Proof.* Since $U$ is bounded, $|\pi_j U| \leq \mathrm{Vol}_{n-1}(\partial U)$ for all $j$. □

## 2. The Sobolev inequality

Let $u \in C^1_{\mathrm{comp}}(\mathbb{R}^n)$; that is, $u$ is a $C^1$ real (or complex) valued function on $\mathbb{R}^n$ with compact support. Given $\int |\nabla u| = 1$, how "big" can $u$ be?

Recall that for $p \geq 1$, the $L^p$ norm is given by

$$\|u\|_{L^p} = \left( \int |u|^p \right)^{1/p}.$$

We'll use the Loomis-Whitney inequality to prove the following important result.

**Theorem 2.1** (Sobolev)**.** $\|u\|_{L^{\frac{n}{n-1}}} \lesssim \|\nabla u\|_{L^1}$.

Before proving this, let's look at a measure of "bigness" related to the $L^{\frac{n}{n-1}}$ norm. Let $S(h) = \{x \in \mathbb{R}^n : |u(x)| > h\}$.

**Proposition 2.2.** *If $\|u\|_{L^p} \leq M$, then $|S(h)| \leq M^p h^{-p}$.*

*Proof.* $M^p \geq \int |u|^p \geq h^p |S(h)|$. □

**Lemma 2.3.** *If $u \in C^1_{comp}(\mathbb{R}^n)$, then $|\pi_j S(h)| \leq h^{-1} \|\nabla u\|_{L^1}$.*

*Proof.* Fix $x \in S(h)$, and look at the line $l$ through $x$ in the $x_j$ direction. Since $u$ is compactly supported, $u = 0$ somewhere on $l$, so $\int_l |\nabla u| \geq h$ by the fundamental theorem of calculus. Thus,

$$\|\nabla u\|_{L^1} \geq \int_{\pi_j S(h) \times \mathbb{R}} |\nabla u| = \int_{\pi_j S(h)} \int_l |\nabla u| \geq |\pi_j S(h)| \cdot h.$$

□

**Corollary 2.4.** $|S(h)| \lesssim h^{-\frac{n}{n-1}} \|\nabla u\|_{L^1}^{\frac{n}{n-1}}$.

*Proof.* This follows from Loomis-Whitney. □

This is close to Theorem 2.1. To actually prove the theorem, we'll need to deal with the possibility of $S(h)$ being small but $u$ growing very quickly within $S(h)$. Luckily, we can do this without too much modification.

*Proof of Sobolev.* Let $S_k = \{x \in \mathbb{R}^n : 2^{k-1} \le |u(x)| \le 2^k\}$.

**Lemma 2.5.** *If $u \in C^1_{comp}(\mathbb{R}^n)$, then $|\pi_j S_k| \lesssim 2^{-k} \int_{S_{k-1}} |\nabla u|$.*

*Proof.* Fix $x \in S_k$, and look at the line $l$ through $x$ in the $x_j$ direction. Since $|u| \le 2^{k-2}$ somewhere on $l$ and $|u| \ge 2^{k-1}$ somewhere on $l$, we have $\int_{S_{k-1}\cap l} |\nabla u| \ge 2^{k-1} - 2^{k-2} = 2^{k-2}$. Thus,

$$\int_{S_{k-1}} |\nabla u| \ge \int_{\pi_j S_{k-1}} \int_{S_{k-1}\cap l} |\nabla u| \ge |\pi_j S_{k-1}| \cdot 2^{k-2}.$$

□

**Corollary 2.6.** $|S_k| \lesssim 2^{-k\cdot\frac{n}{n-1}} \left(\int_{S_{k-1}} |\nabla u|\right)^{\frac{n}{n-1}}$.

*Proof.* This follows from Loomis-Whitney. □

Now, we have

$$\int |u|^{\frac{n}{n-1}} \sim \sum_{k=-\infty}^{\infty} |S_k| \cdot 2^{-k\cdot\frac{n}{n-1}} \lesssim \sum_{k=-\infty}^{\infty} \left(\int_{S_{k-1}} |\nabla u|\right)^{\frac{n}{n-1}} \le \left(\int |\nabla u|\right)^{\frac{n}{n-1}}$$

as desired. □

## 3. $L^p$ estimates of linear operators

Let's do some background in analysis in preparation for the Kakeya problem. Given $f$, $g : \mathbb{R}^n \to \mathbb{R}$ (or $\mathbb{C}$), define the *convolution* $f * g$ by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,dy.$$

For example, define the linear operator $T_\alpha$ by $T_\alpha f = f * |x|^{-\alpha}$, so

$$T_\alpha f(x) = \int_{\mathbb{R}^n} f(y)|x-y|^{-\alpha}\,dy.$$

Assume $0 < \alpha < n$ and $f \in C^0_{\text{comp}}$; then this integral converges for all $x \in \mathbb{R}^n$.

Let's look at $T_\alpha f$ for some specific examples of $f$.

(1) Let $B_1$ denote the unit ball and $\chi_{B_1}$ its characteristic function. It's not hard to see that

$$|T_\alpha \chi_{B_1}(x)| \sim \begin{cases} 1 & \text{if } |x| \le 1, \\ |x|^{-\alpha} & \text{if } |x| > 1. \end{cases}$$

(2) Let $B_r$ denote the ball centered at the origin with radius $r$, $\chi_{B_r}$ its characteristic function. Then

$$|T_\alpha \chi_{B_r}(x)| \sim \begin{cases} r^n r^{-\alpha} & \text{if } |x| \le r, \\ r^n |x|^{-\alpha} & \text{if } |x| > r. \end{cases}$$

We'll leave it at that for examples. Now let's ask the question: For which $p$, $q$ is there an inequality

$$(*) \qquad \|T_\alpha f\|_{L^q} \lesssim \|f\|_{L^p}$$

which holds for all $f$?

If we take $f = \chi_{B_1}$ as in example (1), we have $\|f\|_{L^p} \sim 1$ and $\|T_\alpha f\|_{L^q}^q \sim \int_{\mathbb{R}^n} (1 + |x|)^{-\alpha q}$, which is finite if and only if $\alpha q > n$. So if $(*)$ is to hold, we must have $\alpha q > n$.

If we take $f = \chi_{B_r}$ as in example (2), we have $\|f\|_{L^p} \sim r^{n/p}$. When calculating $\|T_\alpha f\|_{L^q}$, we can show that the "tail" of the integral doesn't contribute too much as long as $\alpha q > n$, so $\|T_\alpha f\|_{L^q} \sim \|r^n r^{-\alpha} \chi_{B_r}\|_{L^q} \sim r^{n-\alpha+n/q}$. So for $(*)$ to hold we need $r^{n-\alpha+n/q} \lesssim r^{n/p}$. Letting $r \to 0$ and $r \to \infty$, we thus need $n - \alpha + n/q = n/p$.