FROM LOCAL TO GLOBAL

In this lecture, we discuss Cayley's theorem on flecnodes and ruled surfaces.

Theorem 0.1. *If P is a polynomial in $\mathbb{C}[z_1, z_2, z_3]$, and if FP vanishes on $Z(P)$, then $Z(P)$ is ruled.*

We know from last lecture that $FP(z) = 0$ if and only if z is flecnodal. So for each $z \in Z(P)$, we know that there is a non-zero vector V so that P vanishes in the direction V to fourth order. Informally, this means that $Z(P)$ locally looks ruled. We want to put the local information together and prove that there are actual global lines contained in $Z(P)$.

Here is the basic difficulty with the proof. Suppose that $V(z)$ is a smooth non-vanishing vector field on $Z(P)$ which obeys the flecnodal equation at each point of $Z(P)$. How can we use V to find lines? A natural method is to look at the integral curves of V. But consider the following example. The surface $Z(P)$ may be a plane. At each point z in the plane $Z(P)$, every tangent vector obeys the flecnodal equation. So let V be any smooth (tangent) vector field in $Z(P)$. It obeys the flecnodal equation at every point, but the integral curves of V are basically arbitrary curves in the plane. If $Z(P)$ is irreducible and not a plane, then this method actually works, but we can see the proof needs to be a little subtle because we need to use the fact that $Z(P)$ is not a plane.

There are also unfortunately a couple of cases in the proof. We won't give a complete proof. Instead we will carefully do one case, which I think of as the main case. Moreover, this one case is enough to give the full proof of the regulus detection lemma.

In our model case, we will work over the real numbers, which is technically easier (and all we need in the regulus detection lemma). The argument works over the complex numbers with minor modifications, but we think it's easier to see the main ideas over $\mathbb{R}$.

Let's recall/clarify our notation for derivatives and higher derivatives, because we will need to be clear-headed about it.

If $F : \mathbb{R}^3 \to \mathbb{R}$ is a function, we write $\partial_i F$ to abbreviate the standard partial derivative $\frac{\partial}{\partial x_i} F$. If V is a vector, we write $\nabla_V F(x)$ for $\sum_i V_i \partial_i F(x)$. The most important role in our story is played by second derivatives. If V, W are two vectors, then we write

$$\nabla^2_{V,W} F(x) = \sum_{i,j} V_i W_j \partial_i \partial_j F(x).$$

We abbreviate $\nabla^2_V = \nabla^2_{V,V}$. Higher derivatives are similar.

Now we can state our special case.

Proposition 0.2. *Suppose that $P \in \mathbb{R}[x_1, x_2, x_3]$. Let $O \subset Z(P)$ be an open subset of $Z(P)$. Suppose that V is a smooth, non-zero vector field on O, obeying the flecnodal equation:*

$$0 = \nabla^s_V P(x), \text{ for all } x \in O, s = 1, 2, 3.$$

We add a technical assumption. Suppose that at each point $x \in O$, $\nabla P(x) \neq 0$ and $\nabla^2 P(x) : TZ \times TZ \to \mathbb{R}$ is non-degenerate.

Then the integral curves of V are straight line segments. Therefore, every point in O lies in a line in $Z(P)$.

A word about the technical assumption. We defined above $\nabla^2_{V,W} P(x)$ for any vectors V, W. Therefore, $\nabla^2 P(x)$ is a map from $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$. We restrict it to a map $TZ \times TZ \to \mathbb{R}$. Being non-degenerate means that for each non-zero $V \in TZ$, there is some $W \in TZ$ so that $\nabla^2_{V,W} P(x) \neq 0$. For most surfaces $Z(P)$, $\nabla^2 P$ is non-degenenerate on a dense open set. In this case, our propisition allows us to find a line of $Z(P)$ thru almost every point. And since the non-degenerate points are dense, we can find a line of $Z(P)$ thru the other points by taking limits. There are, however, some surfaces where $\nabla^2 P$ is degenerate at every point of $Z(P)$. These surfaces require a different argument - so we begin to see that the general theorem requires cases.

Proof. It suffices to show that at each point $x \in O$, $\nabla_V V$ is a multiple of V. If we let V_1 be a unit length renormalization of V, then if follows that $\nabla_{V_1} V_1 = 0$ on O. This equation implies that the integral curves of V_1 (or V) are straight lines.

(Suppose that $\gamma : \mathbb{R} \to O$ is an integral curve of V_1. In other words, $\gamma'(t) = V_1(\gamma(t))$. If we differentiate, we get $\gamma''(t) = \nabla_{V_1(\gamma(t))} V_1(\gamma(t)) = 0$.)

To explain the argument, we need a different derivative - the Lie derivative. If V is a vector field, we let L_V denote the Lie derivative, defined by $L_V F(x) = \sum_i V_i(x) \partial_i F(x)$. Actually, $L_V F(x) = \nabla_V F(x)$, but we come to second derivatives, there is an important difference:

$$L_V(L_V F) \neq \nabla^2_V F! \tag{1}$$

Let's clarify what the left-hand side means. $L_V F$ is a function. Then $L_V(L_V F)$ is the Lie derivative of that function. The reason that the two sides are different is that on the left-hand side, the outer differentiation hits the vector field V appearing in $(L_V F)$. On the right-hand side it doesn't. To compute the right-hand side at a point x, we only need to know V at the point x. But to compute the left-hand side, we need to know V in a small neighborhood - or at least the value of the derivative $\nabla_V V$. This $\nabla_V V$ is a vector field with j^{th} component $= \sum_i V_i \partial_i V_j$. Expanding both sides of (1) and computing, we get:

$$L_V(L_V F) = \nabla_V^2 F + \nabla_{\nabla_V V} F. \tag{2}$$

Now we return to P. We know that $L_V P = \nabla_V P = 0$ on O. Therefore, its derivative vanishes on O, and we get

$$0 = L_V(L_V P) = \nabla_V^2 P + \nabla_{\nabla_V V} P = \nabla_{\nabla_V V} P.$$

So we conclude that $\nabla_{\nabla_V V} P = 0$ on O, and hence $\nabla_V V \in TZ$.

We can get more information by doing a similar computation with third derivatives. A third-order formula analogous to equation (2) reads

$$L_V(\nabla_V^2 F) = \nabla_V^3 F + 2\nabla^2_{\nabla_V V, V} F. \tag{3}$$

We know that $\nabla_V^2 P$ vanishes on O, and therefore its derivative vanishes on O also, and we get:

$$0 = L_V(\nabla_V^2 P) = \nabla_V^3 P + 2\nabla^2_{\nabla_V V, V} P = 2\nabla^2_{\nabla_V V, V} P. \tag{4}$$

So at each point of O, we know that $\nabla_V V \in TZ$ and that $\nabla^2_{\nabla_V V, V} P = 0$. Since we assumed that $\nabla^2 P$ is non-degenerate on TZ, this implies that $\nabla_V V$ is a multiple of V. Here are the details. We assumed that $\nabla^2 P$ is non-degenerate at each point of O. In other words, for each non-zero $v \in TZ$, the kernel of the map $K_v : w \to \nabla^2_{w,v} P$ is one-dimensional. For our particular, V, we know that $\nabla^2_{V,V} P = 0$, and so the kernel of K_V is exactly the span of V. Since $\nabla_V V \in TZ$ and $\nabla^2_{\nabla_V V, V} P = 0$, we conclude that $\nabla_V V$ is in the span of V. $\square$

Exercises and comments. 1. Check that the above argument can be adapted to $\mathbb{C}^3$.

2. The above argument is fundamentally geometric, and it can be adapted to any smooth surface $\Sigma \subset \mathbb{R}^3$. The condition that $\nabla^2 P$ is non-degenerate is equivalent to the second fundamental form of Σ being non-degenerate, which is equivalent to the Gauss curvature of Σ being non-zero.

3. Suppose that $\nabla^2 P$ is degenerate at every point of $Z(P)$. This is equivalent to saying that the Gauss curvature of $Z(P)$ vanishes at every regular point. One example is a cylinder $S^1 \times \mathbb{R}$. In the category of smooth surfaces there are many other examples - take a piece of paper and bend it gently in space. I think there are

also many examples of Gauss flat algebraic surfaces $Z(P)$, but I'm not positive. If $Z(P)$ is Gauss flat and $FP = 0$ on $Z(P)$, prove that $Z(P)$ is still ruled.