

## APPLICATION TO INCIDENCE THEORY OF LINES IN SPACE

In connection with the distinct distance problem, we encountered the following question about lines in  $\mathbb{R}^3$ . Given  $L$  lines in  $\mathbb{R}^3$  with  $\leq L^{1/2}$  lines in any plane, how many points can there be in  $P_k(\mathcal{L})$ ? Elekes and Sharir conjectured the answer to this question, and we will eventually prove their conjecture.

**Theorem 0.1.** *If  $\mathcal{L}$  is a set of  $L$  lines in  $\mathbb{R}^3$  with  $\leq L^{1/2}$  lines in any plane, and  $3 \leq k \leq L^{1/2}$ , then  $|P_k(\mathcal{L})| \lesssim L^{3/2}k^{-2}$ .*

Recall that  $P_k(\mathcal{L})$  is the set of points lying in  $\geq k$  lines of  $\mathcal{L}$ .

Today, we will prove this result for  $k = 3$ . The proof is based on the new techniques we have developed: degree reduction and special points (critical points and flat points).

**Theorem 0.2.** *(Elekes-Kaplan-Sharir) Suppose  $\mathcal{L}$  is a set of  $L$  lines in  $\mathbb{R}^3$  with  $\leq B$  lines in any plane. If  $B \geq L^{1/2}$ , then  $|P_3(\mathcal{L})| \leq CBL$ .*

Taking  $B = L^{1/2}$ , we get the case  $k = 3$  of the conjecture.

As in the proof of the joints theorem, we will try to prove that there is always one line with not too many points of  $P_3$  on it.

**Lemma 0.3.** *Suppose  $\mathcal{L}$  is a set of  $L$  lines in  $\mathbb{R}^3$  with  $\leq B$  lines in any plane, with  $B \geq L^{1/2}$ . Then one of the lines contains  $\leq CB$  points of  $P_3(\mathcal{L})$ .*

Given this lemma, the theorem follows by induction on the number of lines.

### 1. THE PROOF OF THE MAIN LEMMA

We will do a proof by contradiction. We let  $C_0$  be a large constant that we can choose later. We assume that each line of  $\mathcal{L}$  contains  $> C_0B$  points of  $P_3$ .

#### Step 1. Degree reduction

By the degree reduction theorem, there is a non-zero polynomial  $P$  that vanishes on all the lines of  $\mathcal{L}$  with degree  $\lesssim L(C_0B)^{-1}$ . By choosing  $C_0$  sufficiently large, we can assume that

$$d := \deg P \leq (1/100)LB^{-1} \leq (1/100)L^{1/2}.$$

We can also assume that  $P$  is square free.

#### Step 2. The points of $P_3$ are special points for $P$

Recall that a point  $x \in Z(P)$  is called special if it is either critical or flat. Special points can be detected by polynomials. Last lecture, we defined a vector of polynomials  $SP(x)$  so that  $x$  is special if and only if  $P(x) = 0$  and  $SP(x) = 0$ . Each component of  $SP(x)$  has degree  $\leq 3d - 4$ , and there are nine components.

We claim that each point of  $P_3$  is either critical or flat. Let  $x \in P_3$ . By definition,  $x$  lies in (at least) three lines of  $\mathfrak{L}$ . Pick three lines  $l_1, l_2, l_3$  containing  $x$ . We know that  $P = 0$  on these three lines. If the three lines are not coplanar, then we saw in our study of joints that  $x$  is a critical point of  $P$ . The reason is that  $\partial_{v_i} P(x) = 0$  where  $v_i$  are the tangent directions to the three lines. Since these directions form a basis of  $\mathbb{R}^3$ ,  $\nabla P(x) = 0$ . Suppose now that  $x$  is not a critical point of  $P$ . In this case, we will prove that  $x$  is a flat point of  $Z(P)$ .

We see that the lines  $l_1, l_2, l_3$  must lie in a plane. We still have  $\partial_{v_i} P(x) = 0$  for each  $i$ , and so the lines  $l_i$  must all lie in the tangent plane to  $Z(P)$  at  $x$ . Next we perform a translation and rotation so that  $x$  is moved to 0 and  $Z(P)$  is locally described as a graph

$$x_3 = f(x_1, x_2) = Q(x_1, x_2) + O(|x_1| + |x_2|)^3, \quad Q \text{ homogeneous of degree 2.}$$

The translation and rotation moves the lines  $l_i$  to lines  $\tilde{l}_i$  which contain 0 and lie in the plane  $x_3 = 0$ . Since these lines lie in the image of  $Z(P)$ , we see that  $f$  vanishes on them. Therefore,  $Q$  must vanish on them. So we see that the quadratic form  $Q$  vanishes on three lines through 0 in  $\mathbb{R}^2$ . Now by the vanishing lemma,  $Q$  must also vanish on any line that passes through the three lines at distinct points, and it quickly follows that  $Q = 0$ . This means that the point  $x$  is flat.

Each point of  $P_3$  is either critical or flat, and so  $SP(x) = 0$  for each  $x \in P_3$ .

### Step 3. The lines of $\mathfrak{L}$ are special lines for $P$

Each line of  $\mathfrak{L}$  contains  $\geq C_0 B \geq L^{1/2}$  points of  $P_3$ . On the other hand, the polynomial  $P$  has degree  $\leq (1/100)L^{1/2}$ . The polynomials in  $SP$  have degree  $\leq 3d - 4 \leq (3/100)L^{1/2}$ . Each polynomial  $SP$  vanishes at  $\geq L^{1/2}$  points on each line of  $\mathfrak{L}$ . Therefore,  $SP = 0$  on each line of  $\mathfrak{L}$ .

### Step 4. Almost all the lines of $\mathfrak{L}$ must lie in planes of $Z(P)$

Suppose that  $P = \prod_j P_j$  with  $P_j$  irreducible. Some of the  $P_j$  may be linear, and each linear factor corresponds to a plane in  $Z(P)$ . Let  $\pi_1, \dots, \pi_T$  be all the planes in  $Z(P)$ , with  $T \leq d \leq (1/100)L^{1/2}$ . Next we consider how special lines of  $Z(P)$  relate to special lines of the  $Z(P_j)$ .

**Lemma 1.1.** *Suppose that  $l \subset Z(P)$  is a special line, i.e.  $SP = 0$  on  $l$ . Then either  $l$  lies in  $Z(P_j)$  for two different  $j$ , or else  $l$  is a special line of  $Z(P_j)$  for some  $j$ .*

(We note that if  $l$  lies in  $Z(P_j)$  for two different  $j$ , then  $\nabla P = 0$  on  $l$ , so  $l$  is a special line.)

*Proof.* Let  $l$  be a special line of  $Z(P)$ . Suppose that  $l$  lies in  $Z(P_{j_1})$  but  $l$  does not lie in any other  $Z(P_j)$ . We have to show that  $l$  is a special line of  $Z(P_{j_1})$ . First, suppose that  $l$  is a critical line of  $Z(P)$  - in other words,  $\nabla P = 0$  on  $l$ . We expand  $\nabla P$  using the Liebniz formula:

$$\nabla P = \sum_{j_0} (\nabla P_{j_0}) \prod_{j \neq j_0} P_j.$$

Along the line  $l$ ,  $P_{j_1} = 0$ , and so the sum simplifies to  $(\nabla P_{j_1}) \prod_{j \neq j_1} P_j$ . At all but finitely many points of  $l$ , the product does not vanish, and so  $\nabla P_{j_1}$  must vanish. By the vanishing lemma,  $\nabla P_{j_1} = 0$  on  $l$ .

Next, suppose that  $l$  is not a critical line and is a flat line. For almost every point  $x \in l$ ,  $Z(P)$  is a smooth manifold near  $x$  and  $Z(P)$  is flat at  $x$ . But for almost every point  $x$  in  $l$ ,  $Z(P) = Z(P_{j_1})$  in a small neighborhood of  $x$ . So  $Z(P_{j_1})$  is flat at almost every point  $x \in l$ . Therefore,  $SP_{j_1}(x) = 0$  for almost every  $x \in l$ . By the vanishing lemma,  $SP_{j_1} = 0$  on  $l$ .  $\square$

Now we count the number of special lines of various types, using the Bezout theorem. The number of lines that lie in at least two  $Z(P_j)$  is  $\leq \sum_{j_1, j_2} (\deg P_{j_1})(\deg P_{j_2}) \leq d^2 \leq 10^{-4}L$ .

If  $P_j$  is not a linear polynomial, then we proved last time that the number of special lines in  $Z(P_j)$  is  $\leq 3(\deg P_j)^2$ . Therefore, the number of special lines in  $Z(P_j)$  for all the  $P_j$  with  $\deg P_j \geq 2$  is  $\leq 3 \sum_j (\deg P_j)^2 \leq 3d^2 \leq 3 \cdot 10^{-4}L$ .

Therefore, at least  $(99/100)L$  lines of  $\mathfrak{L}$  must lie in the planes  $\pi_1, \dots, \pi_T$ , with  $T \leq d \leq (1/100)L/B$ . By the pigeon hole principle, one of the planes contains at least  $99B$  lines of  $\mathfrak{L}$ . But we assumed that each plane contains  $\leq B$  lines of  $\mathfrak{L}$ . This contradiction proves the lemma.

## 2. WHAT HAPPENS FOR LARGE $k$ ?

We will eventually prove the following theorem.

*If  $\mathfrak{L}$  is a set of  $L$  lines in  $\mathbb{R}^3$  with  $\leq L^{1/2}$  lines in any plane and if  $3 \leq k \leq L^{1/2}$ , then  $|P_k(\mathfrak{L})| \lesssim L^{3/2}k^{-2}$ .*

Suppose that  $\mathfrak{L}$  is a set of  $L$  lines in  $\mathbb{R}^3$  with  $\leq L^{1/2}$  lines in any plane. We would like to show that one of the lines contains  $\leq L^{1/2}k^{-1}$  points of  $P_k(\mathfrak{L})$ . (For heuristic purposes, suppose that each line had  $L^{1/2}k^{-1}$  points of  $P_k$ . Then we would have

$L^{3/2}k^{-1}$  incidences between  $\mathfrak{L}$  and  $P_k$ . But each point of  $P_k$  lies on  $\geq k$  lines, so we would have  $\leq L^{3/2}k^{-2}$  points of  $P_k$ .)

Let us suppose that each line contains  $\geq A$  points of  $P_k$ . For which values of  $A$  can we hope to run the argument above? In order for the argument to work, it's crucial to be able to do degree reduction. Let us do a heuristic calculation of how big  $A$  needs to be to do degree reduction.

Let  $d$  be a degree that we can pick later. We find a non-zero polynomial  $P$  of degree  $\leq d$  that vanishes on  $(1/10)d^2$  random lines of  $\mathfrak{L}$ . Let  $l$  be another line of  $\mathfrak{L}$ . We want to estimate the number of points where this line intersects the random lines above. The line  $l$  contains  $A$  points of  $P_k$ . Each of these points lies in  $\geq k$  lines of  $\mathfrak{L}$ , and so  $l$  intersects  $\geq Ak$  lines of  $\mathfrak{L}$ . Each line of  $\mathfrak{L}$  has a probability  $\sim d^2/L$  of being chosen in the list of random lines. Therefore, the number of lines of  $\mathfrak{L}$  which intersect  $l$  is typically  $Akd^2/L$ . If this number is  $< A$ , then the number of distinct intersection points is typically of the same order of magnitude. The number of intersection points is capped at  $A$ , so we have

$$\mathbb{E}|\{x \in l \mid x \in \text{the } d^2 \text{ random lines}\}| \gtrsim \min(Akd^2L^{-1}, A).$$

We can do degree reduction only if this expected number is  $> d$ . So to do degree reduction we need  $Akd^2L^{-1} > d$  and  $A > d$ . The first inequality is equivalent to  $d > LA^{-1}k^{-1}$ . So to do degree reduction, we need  $A > d > LA^{-1}k^{-1}$  and hence  $A > L^{1/2}k^{-1/2}$ . If  $A$  is, say,  $\geq 10^5 L^{1/2}k^{-1/2}$ , then we can do degree reduction down to degree  $d \sim LA^{-1}k^{-1}$ . The rest of the argument above works. Filling in the details, it is possible to prove the following fairly weak estimate:

**Proposition 2.1.** *Suppose that  $\mathfrak{L}$  is a set of  $L$  lines in  $\mathbb{R}^3$  with  $\leq L^{1/2}$  lines in any plane. Suppose  $3 \leq k \leq L^{1/2}$ . Then one of the lines contains  $\lesssim L^{1/2}k^{-1/2}$  points of  $P_k$ .*

Then, by a somewhat tricky induction argument we get

**Corollary 2.2.** *Suppose that  $\mathfrak{L}$  is a set of  $L$  lines in  $\mathbb{R}^3$  with  $\leq L^{1/2}$  lines in any plane. Suppose  $3 \leq k \leq L^{1/2}$ . Then  $|P_k| \lesssim L^{3/2}k^{-3/2}$ .*

This estimate is significantly weaker than the theorem we will eventually prove. What is the source of our difficulties, and why are we getting stuck at this point?

It's interesting to consider the example of lines in finite fields. Let  $\mathfrak{L}$  be the set of all the lines in  $\mathbb{F}_q^3$ . The total number of lines is  $L \sim q^4$ . The number of lines in any plane is  $\sim q^2 \leq L^{1/2}$ . Each point of  $\mathbb{F}_q^3$  lies in  $k \sim q^2$  lines. Therefore  $|P_k(\mathfrak{L})| = q^3 \sim L^{3/2}k^{-3/2}$ .

This situation is similar to the Szemerédi-Trotter theorem. The Szemerédi-Trotter theorem is true for lines in  $\mathbb{R}^2$ . But it's false over finite fields, in particular if we

consider the set of all lines in  $\mathbb{F}_q^2$ . The known proofs all somehow use the topology of  $\mathbb{R}^2$ . Theorem 0.1 has very similar difficulties.

On the other hand, the case  $k = 3$  of Theorem 0.1 has a nice proof with the polynomial method. This case seems at least as hard as the joints problem, and no one knows how to approach it (so far) without the polynomial method.

So the problem combines the difficulties of the Szemerédi-Trotter theorem and the joints theorem. To prove it, we will need to combine some type of topological argument as in the ST theorem with some type of polynomial argument as in the joints theorem. Our next main goal in the course is to see how to get these two methods to cooperate.

**2.1. On the tricky induction.** Here we describe, for reference, the slightly tricky inductive argument to get the corollary from the proposition. Actually, to do the induction we need a slightly more general proposition.

**Proposition 2.3.** *Suppose that  $3 \leq k \leq 10L^{1/2}$ . Suppose that  $\mathfrak{L}$  is a set of  $L$  lines in  $\mathbb{R}^3$  with  $\leq B$  lines in any plane, with  $B \geq L^{1/2}$ . Then one of the lines of  $\mathfrak{L}$  contains  $\lesssim Bk^{-1/2}$  points of  $P_k$ .*

*Proof sketch.* Suppose that each line contains  $\geq A$  points of  $P_k$ . If  $A \geq 10^5 L^{1/2} k^{-1/2}$ , then we can do degree reduction to fit all the lines in  $Z(P)$  for  $P$  of degree  $\leq CLA^{-1}k^{-1}$ . This degree is  $\leq (1/100)A$  and  $\leq (1/100)L^{1/2}$ . All the points of  $P_k$  are special points of  $Z(P)$ . Since  $A > 3degP$ , all the lines are special lines. Since  $degP \leq (1/100)L^{1/2}$ ,  $Z(P)$  only has room for  $< (1/100)L$  special lines except in the planes of  $Z(P)$ . So almost all the lines actually lie in  $\leq d$  planes. One plane must contain at least  $(1/2)L/d \gtrsim Ak$  lines. Therefore,

$$A \lesssim \min(L^{1/2}k^{-1/2}, Bk^{-1}) \leq Bk^{-1/2}.$$

Using this proposition and induction, we get the following corollary.

**Corollary 2.4.** *Suppose that  $\mathfrak{L}$  is a set of  $L$  lines in  $\mathbb{R}^3$  with  $\leq B$  lines in any plane, where  $B \geq L^{1/2}$ . Suppose that  $3 \leq k \leq L^{1/2}$ . Then  $|P_k| \lesssim LBk^{-3/2}$ .*

*Proof sketch.* Remove the lines one at a time using the last lemma, until we get down to  $(1/100)k^2$  lines. At this point, we know that  $|P_{k/2}| \lesssim k$  by the counting bound. At each step, we remove a line that intersects  $\lesssim LBk^{-1/2}$  points of  $P_{k/2}$ . By the end, all but  $k$  point of  $P_k$  must have had  $k/2$  lines removed. So we see

$$(|P_k(\mathfrak{L})| - Ck)k \lesssim L(Bk^{-1/2}).$$

Hence  $|P_k(\mathfrak{L})| \lesssim LBk^{-3/2}$ .