

### NAMBOODIRI LECTURE 3: INCIDENCE GEOMETRY AND POLYNOMIALS IN FOURIER ANALYSIS

This lecture is about an open problem in Fourier analysis called the restriction problem. At first sight, this problem looks unrelated to the previous lectures. The main goal of the lecture is to understand how they are related – to see how this problem from Fourier analysis relates to geometry, and incidence geometry, and polynomials.

We will be interested in a trigonometric series:

$$f(x) = \sum_{j=1}^M a_j e^{i\omega_j \cdot x}.$$

The problem takes place in  $\mathbb{R}^n$ : the point  $x$  lies in  $\mathbb{R}^n$  and the frequencies  $\omega_j$  lie in  $\mathbb{R}^n$ . The coefficients  $a_j$  are complex numbers. We pick a large radius  $R$ , and we suppose that the frequencies  $\omega_j$  are evenly distributed on the sphere of radius  $R$ , with separation  $\sim 1$  (see Figure 1). The number of different frequencies is then approximately the surface area of the sphere  $S^{n-1}(R)$ , and so we get  $M \sim R^{n-1}$ . We also suppose that all the coefficients  $a_j$  have norm  $|a_j| = 1$ .

Under all these assumptions, we would like to estimate the moments of  $f$  on the unit cube  $Q = [0, 1]^n$ .

**Question 1.** *If  $f(x)$  is a trigonometric series as above, then estimate the maximum possible size of the integral*

$$\int_Q |f|^p,$$

(in terms of  $R$  and  $p$ ).

We will spend a good portion of this lecture just processing this question, trying to understand what it means, and why it is interesting and difficult.

**0.1. Visualizing the problem.** Let us start by visualizing a single complex exponential  $a_1 e^{i\omega_1 \cdot x}$ . We consider the set of points  $x$  where this complex exponential takes the value 1. Since  $|a_1| = 1$ , we note that

$$a_1 e^{i\omega_1 \cdot x} = 1 \leftrightarrow \omega_1 \cdot x \in 2\pi\mathbb{Z} - \text{phase}(a_1).$$

So the set where  $a_1 e^{i\omega_1 \cdot x} = 1$  is a union of parallel hyperplanes, perpendicular to the vector  $\omega_1$ . The spacing between consecutive hyperplanes is  $\frac{2\pi}{R}$ . The function  $a_1 e^{i\omega_1 \cdot x}$  looks like a wave: if we consider the real part, it has its peaks on the parallel hyperplanes described above, and it has its troughs half way between two consecutive peaks.

Next let us visualize a second complex exponential, and think about how it interacts with the first complex exponential. The set where  $a_2 e^{i\omega_2 \cdot x} = 1$  is a union of parallel hyperplanes, perpendicular to the vector  $\omega_2$ . We illustrate the situation in Figure 2. The blue lines represent the hyperplanes where  $a_1 e^{i\omega_1 \cdot x} = 1$  and the green lines represent the hyperplanes where  $a_2 e^{i\omega_2 \cdot x} = 1$ . We want to understand the sum  $a_1 e^{i\omega_1 \cdot x} + a_2 e^{i\omega_2 \cdot x}$ . On the shaded dots, we have constructive interference: both

terms are 1 or both terms are -1. At the  $x$ 's, we have cancellation: one term is 1 and the other is -1. From this picture we can get a sense of how these two complex exponentials interact.

Now the reader should imagine adding a third complex exponential  $a_3 e^{i\omega_3 x}$ . When we extend the picture, we will see a third color of hyperplanes oriented in a third direction. The pattern of constructive interference vs. cancellation will become quite complicated. The trigonometric sum  $f$  is a sum of a huge number of complex exponentials. To understand Question 1, we have to understand how much constructive interference there is and how much cancellation there is in a sum of many waves.

Understanding constructive interference vs. cancellation is a central theme of Fourier analysis which comes up in many problems in both pure and applied math. Question 1 is one particular problem about constructive interference vs. cancellation, but it has been studied very intensively, and the ideas that have come out of this work have had applications in many other scenarios.

**0.2. A random model for comparison.** To get a perspective, let us consider what would happen if we replaced the complex exponentials  $e^{i\omega_j x}$  with *independent* functions  $h_j(x)$ . Here we mean independent in the sense of probability theory - as in the phrase independent random variables. Suppose that each function  $h_j$  goes from the unit cube  $Q$  to the unit complex numbers, and suppose that the values of each  $h_j$  are evenly distributed on the unit circle. In this independent case, the behavior of the sum  $\sum a_j h_j$  is governed by the law of large numbers. The law of large numbers implies that with high probability,  $\left| \sum_{j=1}^M a_j h_j(x) \right|$  has size on the order of  $M^{1/2}$ . More precisely, we can say that

$$\left| \sum_{j=1}^M a_j h_j(x) \right| \leq KM^{1/2},$$

except for a small set of points  $x$  whose measure decays exponentially in  $K$ . As a consequence, for any exponent  $1 \leq p < \infty$ , we get

$$\int_Q \left| \sum_{j=1}^M a_j h_j(x) \right|^p dx \leq C_p M^{p/2}. \quad (1)$$

We refer to this behavior as square-root cancellation.

This random model gives us some perspective on our original problem. We want to understand how close the functions  $e^{i\omega_j x}$  are to being independent of each other. What are the strongest, most important correlations among these functions?

**0.3. An example: all coefficients are 1.** Now we return to trigonometric sums  $\sum_{j=1}^M a_j e^{i\omega_j x}$ , and we consider the example that all the coefficients  $a_j$  are 1. If  $|x| \leq \frac{1}{10R}$ , then  $e^{i\omega_j x}$  is close to 1 for all of our frequencies  $\omega_j$ . Since  $a_j = 1$  for all  $j$ , we get pure constructive interference on the ball defined by  $|x| \leq \frac{1}{10R}$ : in other words,

$$|f(x)| \sim M \text{ if } |x| < \frac{1}{10R}.$$

This ball has volume  $\sim R^{-n}$ , and so we get a lower bound on the moments of  $f$ :

$$\int_Q |f|^p \gtrsim R^{-n} M^p.$$

In particular, if  $p$  is large enough (computed below), then  $R^{-n}M^p$  is much larger than  $M^{p/2}$ , and we don't get square-root cancellation.

0.4. **The restriction conjecture.** In the late 1960's, Stein conjectured that the moments of  $f$  are always bounded either by square-root cancellation or by the last example.

**Conjecture 2.** (Stein) If  $f = \sum_{j=1}^M a_j e^{i\omega_j x}$  with frequencies  $\omega_j$  as in the introduction and  $|a_j| = 1$ , then

$$\int_Q |f|^p \lesssim M^{p/2} + R^{-n} M^p.$$

(Remark. We write  $A \lesssim B$  to mean either  $A \leq CB$  or  $A \leq C(\log R)^{\text{power}} B$ . In this conjecture, we actually need a logarithmic factor for some values of  $p$ . In Section 3, we will describe an example where the logarithmic powers are needed.)

Plugging in  $M = R^{n-1}$ , we can check when each term dominates. If  $p \leq \frac{2n}{n-1}$ , then the first term dominates and we have square root cancellation like for independent functions. If  $p > \frac{2n}{n-1}$ , then the second term dominates, and we don't have square root cancellation. The critical exponent  $p = \frac{2n}{n-1}$  is the most important.

If this conjecture is true, it means that the pure constructive interference on the ball  $B(\frac{1}{10R})$  is the strongest, most important correlation among the functions  $e^{i\omega_j x}$ . If this conjecture is false, it means that there is some other, subtler correlation which is even stronger (at least for some exponents  $p$ ). Either way, it would be very interesting to know.

Here is a brief summary of known results. In dimension  $n = 2$ , the conjecture was proven by Fefferman in the 1970's. For dimensions  $n \geq 3$ , the conjecture is open in spite of a lot of effort by many people. It is true for  $p = 2$  by an easy argument – see the next section. It is also true if  $p$  is sufficiently large (depending on the dimension  $n$ ). For  $n = 3$ , the best current result says that the theorem is true if  $p > 3.25$  (see [G1]). The proof uses the polynomial method, as well as many earlier ideas. In the rest of the lecture, we will describe some of these ideas.

The notes [T] are a great reference on the restriction problem. (I should probably mention here that the restriction problem is usually phrased in terms of an integral over all the frequencies on the sphere instead of a sum over a separated set of frequencies. The two versions of the problem are equivalent, and I felt that the trigonometric is probably more accessible for a broad audience.)

### 1. ORTHOGONALITY

Now we begin to discuss how to prove bounds for our integral  $\int_Q |f|^p$ . Our first bounds will be based on the observation that the functions  $\{e^{i\omega_j x}\}$  are (almost) orthogonal on the unit cube  $Q$ . By orthogonality, we see that

$$\int_Q |f|^2 \sim \sum_{j=1}^M \int_Q |e^{i\omega_j x}|^2 = M.$$

This shows that Conjecture 2 is true for  $p = 2$ . (Holder's inequality then implies the conjecture is true for all  $p \leq 2$ . The interest is in  $p > 2$ .)

Also, the triangle inequality gives a simple bound for the supremum of  $f$ :

$$\sup_x |f(x)| = \sup_x \left| \sum_{j=1}^M a_j e^{i\omega_j x} \right| \leq M.$$

Together, these two bounds give an upper bound for  $\int_Q |f|^p$  for any  $p \geq 2$ .

**Proposition 3.** *If  $p \geq 2$ ,*

$$\int_Q |f|^p \lesssim M^{p-1}.$$

*Proof.*

$$\int_Q |f|^p \leq (\sup |f|)^{p-2} \int_Q |f|^2 \lesssim M^{p-2} M.$$

□

The bound  $M^{p-1}$  is bigger than the bound in Conjecture 2. We will refer to this argument as “the orthogonality argument” because it basically boils down to orthogonality.

**1.1. The sphere vs. the disk.** Our frequencies  $\omega_j$  are evenly distributed on the sphere of radius  $R$ . For comparison, suppose that we consider frequencies  $\tilde{\omega}_j$  evenly distributed on an  $(n-1)$ -dimensional disk of radius  $R$  – see Figure 3. We suppose that  $\tilde{\omega}_j$  are a 1-separated set of frequencies that lie in the hyperplane with  $n^{\text{th}}$  coordinate zero, and with  $|\tilde{\omega}_j| \leq R$ . The number of frequencies  $\tilde{\omega}_j$  is still  $\sim R^{n-1}$ , so we can still let  $j$  go from 1 to  $M$ . We let  $g$  be a trigonometric sum using frequencies on the disk:

$$g(x) = \sum_{j=1}^M a_j e^{i\tilde{\omega}_j x}.$$

The functions  $e^{i\tilde{\omega}_j x}$  are again (almost) orthogonal on the unit cube, so the orthogonality argument applies to  $g$ , just as well as it did to  $f$ . Hence  $\int_Q |g|^p \lesssim M^{p-1}$ . This bound for  $g$  is actually sharp when all the coefficients  $a_j$  are 1!

We can see this using a key feature of the disk setup: the functions  $e^{i\tilde{\omega}_j x}$  don’t depend on the  $x_n$  coordinate. Because of this observation, all the functions  $e^{i\tilde{\omega}_j x}$  are close to 1 when the first  $n-1$  coordinates of  $x$  lie in  $B^{n-1}(\frac{1}{10R})$ . In other words,

$$\left| \sum_j e^{i\tilde{\omega}_j x} \right| \sim M \text{ for } x \in B^{n-1}\left(\frac{1}{10R}\right) \times [0, 1].$$

Now a simple calculation shows that  $\int_Q |g|^p \gtrsim M^{p-1}$  for all  $p \geq 2$ .

Recall for comparison that in the sphere setup, the sum  $\sum_j e^{i\omega_j x}$  has pure constructive interference in the ball  $B^n(\frac{1}{10R})$ . The region  $B^{n-1}(\frac{1}{10R}) \times [0, 1]$  is significantly larger than the ball  $B^n(\frac{1}{10R})$  – see Figure 4 – which makes the disk behave very differently from the sphere.

Stein realized that the sphere behaves differently from the disk. He proved estimates for the sphere that improve on the orthogonality argument, estimates that hold for the sphere but don’t hold for the disk. His arguments show how the curvature of the sphere relates to estimates for  $\int |f|^p$ . This connection between the geometric idea of curvature and Fourier analysis is an exciting

discovery, and the full strength of the connection is not yet understood today because Conjecture 2 is still wide open.

In the next sections, we will explain one way that the curvature enters into the estimates.

## 2. THE CONTRIBUTION FROM A SMALL CAP

We want to understand the behavior of the sum  $f = \sum_j a_j e^{i\omega_j x}$  on the unit cube. Understanding the whole sum is difficult, so it is reasonable to warm up by trying to understand just a small piece of the sum. Let  $\theta \subset S^{n-1}(R)$  be a spherical cap of some radius much smaller than  $R$ . We let  $f_\theta$  denote the contribution to  $f$  from the frequencies in  $\theta$ :

$$f_\theta(x) := \sum_{\omega_j \in \theta} a_j e^{i\omega_j x}.$$

If  $\theta$  is small enough, can we get a good understanding of  $f_\theta$  on the unit cube? It turns out that if  $\theta$  is a cap of radius  $\frac{1}{100}R^{1/2}$ , then we have a very good understanding of the behavior of  $f_\theta$  on the unit cube. The reason is that for this radius, the cap  $\theta$  lies very close to a hyperplane. The cap  $\theta$  lies inside of a rectangular slab with one short direction of length  $\frac{1}{10}$  and  $n - 1$  long directions of length  $\sim R^{1/2}$ : see Figure 5. The radius  $\frac{1}{100}R^{1/2}$  was chosen just so that the thickness of the slab would be significantly less than 1.

If  $|\omega - \omega'| < (1/10)$ , then the functions  $e^{i\omega x}$  and  $e^{i\omega' x}$  look quite similar on the unit cube. Therefore,  $f_\theta$  behaves almost as if all the frequencies  $\omega_j \in \theta$  were contained in a hyperplane. The function  $f_\theta$  behaves a lot like the function  $g$  in the disk problem from the last subsection. This leads to some important consequences about the function  $f_\theta$ .

- (1) The orthogonality argument gives sharp estimates for  $\int_Q |f_\theta|^p$  for any  $p$ .
- (2)  $|f_\theta(x)|$  is essentially constant on line segments perpendicular to  $\theta$  and contained in  $Q$ .

The second item is analogous to the fact that the function  $g(x)$  did not depend on the  $x_n$  coordinate. Figure 6 shows a picture of the cap  $\theta$  and a line segment in the unit cube which is perpendicular to  $\theta$ . The norm  $|f_\theta(x)|$  is morally almost constant along this green line segment.

Let us use this information to try to visualize  $f_\theta$ . To help with the visualization, we pick some threshold  $\lambda > 0$ , and we color a point  $x \in Q$  green if  $|f_\theta(x)| > \lambda$ . Since  $|f_\theta(x)|$  doesn't change much when we move in the direction perpendicular to  $\theta$ , the green set should be a union of line segments that are perpendicular to  $\theta$ . A more careful analysis shows that the green set should be a union of green cylinders of radius  $\sim R^{-1/2}$  and length  $\sim 1$ , with their long axis perpendicular to  $\theta$ . Figure 7 shows a picture of the cap  $\theta$  and the green set inside of  $Q$ .

There are a range of possible behaviors of  $|f_\theta|$ . At one extreme,  $|f_\theta|$  can be large in a single cylinder of radius  $\sim R^{-1/2}$  and small everywhere else. This is the most concentrated behavior of  $|f_\theta|$ : in this case  $\max |f_\theta|$  is the largest, and it also gives the largest value of  $\int_Q |f_\theta|^p$  for any  $p > 2$ . At the other extreme,  $|f_\theta|$  can be essentially constant on the whole cube  $Q$ . This leads to the smallest value of  $\int_Q |f|^p$ . And we can have any intermediate behavior:  $|f_\theta|$  can concentrate in any number of green cylinders of radius  $R^{-1/2}$ . Figure 8 illustrates this range of behaviors.

Now that we understand  $f_\theta$  for a single cap  $\theta$  of radius  $\sim R^{1/2}$ , we have to understand how the contributions from different caps interact. We cut the sphere  $S^{n-1}(R)$  into disjoint pieces  $\theta$  which are each roughly spherical caps of radius  $\sim \frac{1}{100}R^{1/2}$ . Then our original function  $f$  is a sum of contributions from the caps:

$$f(x) = \sum_{\theta} f_{\theta}(x).$$

Each function  $f_{\theta}$  behaves in the way we described above. In Figure 9, we show  $S^{n-1}(R)$  divided into several caps, which we draw in different colors. Then we show the corresponding functions  $f_{\theta}$  in the unit cube. In the unit cube, we draw a set of green cylinders which mark the region where the contribution from the green cap is large, a set of blue cylinders that mark the region where the contribution from the blue cap is large, etc. Note that the different colored cylinders point in different directions. This has to do with the curvature of the sphere. The blue cylinders point in the direction normal to the blue cap, and because of the curvature of the sphere, each cap has a distinct normal direction.

Figure 9 is one of my favorite pictures in Fourier analysis, and explaining it is one of the main goals of this lecture. The picture is supposed to help visualize how the functions  $f_{\theta}$  are related to each other. In the next section, we begin to discuss how this setup can be used to bound  $\int_Q |\sum_{\theta} f_{\theta}|^p$ .

### 3. INTERSECTION PATTERNS OF TUBES IN FOURIER ANALYSIS

In this section we start to explore how the intersection pattern of the tubes in Figure 9 is related to the size of  $\int_Q |f|^p$ . To get started, we consider the very special case that all these tubes are disjoint. In other words, we suppose that the functions  $f_{\theta}$  have disjoint support. In this special case, we have

$$\int_Q |f|^p = \sum_{\theta} \int_Q |f_{\theta}|^p.$$

Recall from the last section that we have a good understanding of each function  $f_{\theta}$ . In particular, we have a sharp upper bound for  $\int_Q |f_{\theta}|^p$ . Plugging this bound into the right-hand side, we get a sharp upper bound for  $\int_Q |f|^p$ , proving Stein's conjecture in the case when  $f_{\theta}$  have disjoint supports.

Here is a stranger example. Suppose that Figure 9 has only one tube of each color – exactly one tube for each cap  $\theta$ . If we choose the tubes so that they all go through the center of  $Q$ , then the tubes will cover most of  $Q$  and a typical point in their union will lie in  $\sim 1$  tube. In the 1920's, Besicovitch constructed a way to arrange the tubes (still one tube for each cap  $\theta$ ) so that their union has area  $\sim (\log R)^{-1}$ , and a typical point in the union lies in  $\sim \log R$  tubes. There are nice pictures of this arrangement of tubes in the wikipedia article on Besicovitch sets, among other places. In the early 70's, Fefferman used Besicovitch's construction to build a function  $f$  where  $\int_Q |f|^p$  is on the order of  $(\log R)^{\text{power}} M^{p/2}$ . This example came as a great surprise to people. It shows that the logarithmic factor is necessary in the restriction conjecture. It is an important open problem whether the factor  $\log R$  in Besicovitch's example can be increased to a power of  $R$ . This problem is called the Kakeya problem, and it is closely related to the restriction problem.

So the main worry is that the tubes in Figure 9 have many rich intersections. Most of the work on the restriction problem proceeds by studying the intersection pattern of the tubes, using geometry and combinatorics to estimate the number of rich intersections, and using this estimate to help bound  $\int_Q |f|^p$ . This strategy goes back to Fefferman and Cordoba in the 1970's, but it was greatly generalized by Bourgain in the early 1990's. Using this method, he was able to prove deep new estimates about the restriction problem in dimension  $n \geq 3$ , breaking new ground in a direction

where there had been no progress in a long time. This development is very well described in the notes [T].

In the late 1990's, Tom Wolff made the analogy between this problem about rich intersections of thin tubes and problems in incidence geometry like the Szemerédi-Trotter problem. To help make the analogy clear, let's recall the setup of the Szemerédi-Trotter theorem from Lecture 1.

We let  $\mathfrak{L}$  denote a set of lines.

$$P_r(\mathfrak{L}) := \{x \in \mathbb{R}^2 \text{ so that } x \text{ lies in at least } r \text{ lines of } \mathfrak{L}\}.$$

$$P_r(L) := \max_{|\mathfrak{L}|=L} |P_r(\mathfrak{L})|.$$

The Szemerédi-Trotter theorem estimates  $P_r(L)$  up to a constant factor. Our current situation has a similar flavor. We have a bunch of thin tubes pointing in different directions, and we want to understand how many  $r$ -rich intersections there can be.

At the time Wolff started thinking in this direction, incidence geometry and Fourier analysis were very separate communities. Wolff learned the ideas from this other field, especially the cutting method from [CEGSW] that we discussed in Lecture 1, and he applied them to prove new hard theorems in Fourier analysis (cf. [W1] and [W2]). Ever since then, the two fields have been closely connected and new ideas in one field often lead to some progress in the other.

In particular, the polynomial partitioning method that we discussed in Lecture 2 has led to some small progress on the restriction conjecture in three dimensions. In Lecture 2, we discussed using polynomial partitioning to estimate the number of  $r$ -rich intersection points of lines in  $\mathbb{R}^3$ . That argument can be adapted to study the restriction conjecture in  $\mathbb{R}^3$ . In the rest of the lecture we describe how to set up the argument, following [G1].

#### 4. POLYNOMIAL PARTITIONING AND RESTRICTION

Recall that  $\text{Poly}_D(\mathbb{R}^3)$ , the space of polynomials of degree at most  $D$  on  $\mathbb{R}^3$ , has dimension  $\sim D^3$ . Therefore, based on parameter counting considerations, it is reasonable to try to find a polynomial that divides something into  $\sim D^3$  equal pieces. Using the partitioning argument from [GK], it follows that for any  $D \geq 1$ , there is a non-zero polynomial  $P \in \text{Poly}_D(\mathbb{R}^3)$  so that  $Q \setminus Z(P)$  is a disjoint union of  $\sim D^3$  open cells  $O_i$ , and so that for each  $O_i$ ,

$$\int_{O_i} |f|^p = \frac{1}{\# \text{ of cells}} \int_Q |f|^p.$$

Figure 10 shows a picture of  $Z(P)$  inside of  $Q$ . The zero set  $Z(P)$  is shown in red, over the background of the blue and green cylinders which show the regions where different  $f_\theta$  are big.

A line can enter at most  $D + 1$  of the cells  $O_i$ , because if  $P$  vanishes at more than  $D$  points of a line, then the whole line must lie in  $Z(P)$ . This fact played a crucial role in the application of polynomial partitioning to incidence geometry, which we talked about at the end of Lecture 2.

In the setting of Fourier analysis, we are dealing not with perfect lines but with thin cylinders of radius  $R^{-1/2}$ . Is it true that a thin cylinder can also enter only  $D + 1$  cells? No, a thin cylinder can enter far more cells. We tried to give some sense of this in Figure 10. In the figure, there are four cells. Any line enters at most three cells, but the purple cylinder enters all four cells. In fact, the purple cylinder contains the central singular point of  $Z(P)$ , and even a tiny neighborhood of this singular point intersects all four cells.

To deal with this issue, we separate  $Q$  into a part near  $Z(P)$  and parts far from  $Z(P)$ . Let  $W$  be the  $R^{-1/2}$ -neighborhood of  $Z(P)$ , and let us recall that  $R^{-1/2}$  is the radius of our thin cylinders. We let  $O'_i = O_i \setminus W$ , the part of  $O_i$  which is not too close to  $Z(P)$ . We call  $O'_i$  a shaved cell.

**Lemma 4.** *Each cylinder  $T$  of radius  $R^{-1/2}$  enters at most  $D + 1$  of the shaved cells  $O'_i$ .*

Proof idea. Suppose that  $T$  is the  $R^{-1/2}$  neighborhood of a line  $\ell$ . We know that  $\ell$  enters at most  $D + 1$  cells  $O_i$ . So we just have to check that if  $T$  enters a shaved cell  $O'_i$ , then  $\ell$  enters the original cell  $O_i$ . Indeed, suppose that  $x' \in T \cap O'_i$ . Since  $x'$  is not in  $W$ , we know that the distance from  $x'$  to the boundary of  $O_i$  is at least  $R^{-1/2}$ . But the distance from  $x'$  to the line  $\ell$  is at most  $R^{-1/2}$ , and so there must be a point  $x \in \ell$  which lies in  $O_i$ .

We can now very roughly sketch the method of bounding  $\int_Q |f|^p$ . We divide the integral into a contribution from  $W$ , and contributions from the shaved cells  $O'_i$ .

$$\int_Q |f|^p = \int_W |f|^p + \sum_i \int_{O'_i} |f|^p.$$

If the contribution of  $W$  dominates, it is possible to bound the problem using 2-dimensional methods. Recall that the restriction conjecture was proven in two dimensions in the 1970's. The region  $W$  is a narrow neighborhood of the two-dimensional surface  $Z(P)$ , and the same methods that work to study a 2-dimensional cube can be adapted to give good estimates for the contribution of  $W$ .

If the contribution of the cells dominates, then we can study the problem using induction. In this case, the total integral is approximately  $D^3$  times the contribution of a typical shaved cell  $O'_i$ . We know that each tube only enters  $\sim D$  out of the  $\sim D^3$  cells. So a typical cell intersects a fraction  $\sim \frac{1}{D^2}$  of the tubes. By induction on the number of tubes, we can get an estimate for the contribution of a typical cell, which then gives a bound for  $\int_Q |f|^p$ . With the current technology, this inductive argument closes for  $p > 3.25$ .

## 5. CONCLUSION AND FUTURE DIRECTIONS

In hindsight, the main character in these lectures was the method of polynomial partitioning. Polynomial partitioning gives a nice proof of the Szemerédi-Trotter theorem, one of the main results discussed in the first lecture. It plays a major role in the proof of the distinct distance conjecture, the subject of the second lecture. And it is also useful for studying the restriction conjecture in Fourier analysis, which looks at first to be a very different subject.

The polynomial partitioning argument is based on two fundamental facts about polynomials, which we emphasized in the first lecture:

- The dimension of  $\text{Poly}_D(\mathbb{R}^n) \sim D^n$ .
- If  $P \in \text{Poly}_D(\mathbb{R}^n)$  vanishes at  $D + 1$  points on a line  $\ell$ , then  $P$  vanishes on  $\ell$ .

These two simple facts belong to the undergraduate algebra curriculum. They work well together as a team, and I think it's remarkable how many consequences they have. In these lectures, we talked about applications in incidence geometry and in Fourier analysis. There are also other applications in computer science and in differential geometry (see section 2 of [G2] for an introduction to these directions).

I would like to end by talking a little about future directions. The argument that we briefly sketched proves the restriction conjecture in  $\mathbb{R}^3$  in a certain range of values of  $p$ :  $p > 3.25$ . This



result slightly improved the known range of exponents  $p$ , but it is far from solving the full problem. I will try to explain where the argument was not sharp and this will point to some future directions.

If we unwind the inductive argument, we can describe a hypothetical example which is the worst-case scenario in our argument. This hypothetical example probably doesn't exist, and if it did it would be a counterexample to the restriction conjecture. In this hypothetical example, at the first step of the argument, the cells dominate. We examine each cell using the same argument, so we subdivide each cell into smaller cells. We subdivide for some number of steps, say  $s$  steps, and we now have  $D^{3s}$  small cells, and each tube enters  $\sim D^s$  of the small cells. Now when we examine each of these small cells, the contribution to  $\int |f|^p$  is concentrated near an algebraic surface (analogous to the region  $W$ ). This algebraic surface may well be a plane, and so in each small cell,  $\int |f|^p$  is dominated by the contribution from a thin slab. Each of these thin slabs is criss-crossed with the tubes from Figure 9. Also, the different slabs need to face in different directions. See Figure 11 for a rough picture of this hypothetical example.

For any individual cell, we can describe the worst-case arrangement of tubes in that cell. Our bound comes from taking the worst-case scenario in each cell and adding them up. But each tube actually enters many cells, and it seems unlikely that the worst-case scenario can really occur in every cell. If I try to build such an example, here is what happens. I can cleverly arrange the tubes to give the worst-case scenario in a particular cell, call it Cell 1. Now each of these tubes that goes through Cell 1 intersects many other cells. Consider a particular tube  $T$ , and let Cell 2 be one of the other cells that  $T$  enters. If I try to arrange the tubes to achieve the worst-case scenario in Cell 2, I find that the tube  $T$  is not in the right place. If I move  $T$  to get the worst-case scenario in Cell 2, then it is no longer in the right position to give the worst-case scenario in Cell 1. It's very difficult - and probably impossible - to fit together the puzzle pieces to get the worst-case scenario in every cell. But it's hard to prove anything about this - it's hard to keep track of how what is going on in one cell is related to what is going on in the other cells.

This issue of keeping track of the relationships between the different cells comes up in both Fourier analysis and incidence geometry. It comes up in all of the problems we've discussed in these lectures. To give one final example, it comes up in studying the structure of nearly sharp examples for the Szemerédi-Trotter theorem.

Recall from the beginning of Lecture 2 that for a large range of the parameters, all the known roughly sharp examples for the Szemerédi-Trotter theorem are integer grids. It seems plausible that all the roughly sharp examples really are integer grids, and it seems likely that they have a lot of algebraic structure in some sense. But we can prove almost nothing about the structure of near-sharp examples. It is a funny situation. Just counting up what happens in each cell and adding gives sharp bounds for  $|P_r(\mathcal{L})|$ , but it tells us nothing about examples that are close to these sharp bounds. The structural properties of near-sharp examples have to do with the relationships between what is happening in different cells. Understanding these relationships better is a major issue in the field.

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