NAMBOODIRI LECTURE 1: INTRODUCTION TO THE POLYNOMIAL METHOD AND INCIDENCE GEOMETRY

Thank you for the invitation to give these lectures. I'm excited to be here. I want to tell you about some applications of polynomials to problems in combinatorics and Fourier analysis. What I think is interesting here is that the statements of the problems do not involve polynomials, but it turns out that polynomials provide a crucial structure under the surface.

I learned about this technique from a paper by Zeev Dvir in 2008. Dvir [D] proved a conjecture called the finite field Kakeya conjecture on the borderline between combinatorics and harmonic analysis. People in the field had believed that this problem was a really hard conjecture, but the proof is only two pages long, and it only requires an undergraduate background to understand. The proof uses a trick with polynomials, which is related to ideas from error-correcting codes, one of Zeev's areas of study. Since then, a number of mathematicians have been exploring what else this trick can do. So my goal for these lectures is to explain how this trick works, look at some applications, and think about how it fits into mathematics.

I am currently working on a book which will explain the topics of these lectures in a lot more detail. I hope that it will be ready in about a year.

1. The joints problem

Let us look at a short proof using this polynomial trick. Our first proof solves a problem about lines in $\mathbb{R}^3$ called the joints problem.

$$\text{Let } \mathfrak{L} \text{ denote a set of } L \text{ lines in } \mathbb{R}^3.$$

Definition 1. *A point $x \in \mathbb{R}^3$ is a joint of $\mathfrak{L}$ if x lies in three lines of $\mathfrak{L}$ that are not coplanar.*

$$J(\mathfrak{L}) := \{x \in \mathbb{R}^3 | x \textit{ lies in three lines of } \mathfrak{L} \textit{ that are not coplanar}\}.$$

The joints problem is to estimate

$$\max_{|\mathfrak{L}|=L} |J(\mathfrak{L})|.$$

For example, we consider a 3D grid.

$$\mathfrak{L} := \{\text{Axis parallel lines thru an } S \times S \times S \text{ grid of points}\}.$$

Each point of the grid is a joint of $\mathfrak{L}$, so $|J(\mathfrak{L})| = S^3$. There are S^2 lines of $\mathfrak{L}$ in each coordinate direction, and so $L = 3S^2$. Therefore, in this example,

$$|J(\mathfrak{L})| \sim L^{3/2}.$$

In the early 90's, [CEGPSSS] proposed the joints problem and raised the question whether the exponent 3/2 is optimal. There were a sequence of interesting partial results, but a sharp estimate seemed very difficult before the polynomial method. Now we can give a two page proof.

Theorem 1. $|J(\mathfrak{L})| \leq 10 L^{3/2}$.

(The key idea in the proof comes from [D]. The theorem was first proven in [GK], and it was simplified and generalized in [KSS] and [Q].)

The main lemma in the proof says that there is always a line $\ell_0 \in \mathfrak{L}$ that does not contain too many joints.

Main Lemma. *There exists $\ell_0 \in \mathfrak{L}$ so that*

$$|J(\mathfrak{L}) \cap \ell_0| \leq 3|J(\mathfrak{L})|^{1/3}.$$

For comparison, in the grid example, every line contains $S = |J(\mathfrak{L})|^{1/3}$ joints.

Proof. We let P be a minimal degree (non-zero) polynomial that vanishes on $J(\mathfrak{L})$.

Claim 1. There exists a line $\ell_0 \in \mathfrak{L}$ where P does not vanish identically.

We prove claim 1 by contradiction: suppose that P vanishes on every line of $\mathfrak{L}$. Consider a joint $x \in J(\mathfrak{L})$. The point x lies in three lines $\ell_1, \ell_2, \ell_3 \in \mathfrak{L}$. Let v_i be a tangent vector to ℓ_i. Since P vanishes along ℓ_i, we see that the derivative of P at x must vanish in the direction of ℓ_i:

$$\nabla P(x) \cdot v_i = 0 \text{ for all } i.$$

Since the lines ℓ_1, ℓ_2, ℓ_3 are not coplanar, the vectors v_i form a basis of $\mathbb{R}^3$. Therefore, we conclude that $\nabla P(x) = 0$. In other words, the three partial derivatives $\partial_1 P$, $\partial_2 P$, and $\partial_3 P$ must vanish on $J(\mathfrak{L})$. The degree of $\partial_i P$ is strictly smaller than the degree of P. Since P is a non-zero polynomial, these partial derivatives are not all zero, and we arrive at a contradiction: a polynomial of lower degree that vanishes on $J(\mathfrak{L})$.

To prove the main lemma, we will show that the line ℓ_0 from Claim 1 contains few joints. The polynomial P has helped us to find this line. The intuition is that P vanishes on the "important" lines in the configuration but not on the unimportant lines. For example, consider a 2×3 grid of lines in the plane, as in Figure 1. (The figures for each lecture are in a separate file on my webpage.) Notice that there are two horizontal lines that go through three points in the grid, and some vertical lines that go through only two points. The lowest degree polynomial that vanishes on these six points has degree 2: its zero set consists of the two horizontal lines. So the lowest degree polynomial vanishes on the lines containing three points of the grid, but not on the lines containing two points of the grid.

Next we have to estimate the number of joints on ℓ_0.

Claim 2. If P does not vanish on ℓ_0, then $|J(\mathfrak{L}) \cap \ell_0| \leq \operatorname{Deg} P$.

This follows immediately from a fundamental fact about polynomials:

Vanishing Lemma. *If P vanishes at $> \operatorname{Deg} P$ points on a line ℓ, then $P|_\ell = 0$.*

So to bound the number of joints on ℓ_0, it just remains to estimate $\operatorname{Deg} P$.

Claim 3. $\operatorname{Deg} P \leq 3|J(\mathfrak{L})|^{1/3}$.

To prove this claim, we have to show that there is some non-zero polynomial of degree at most $3|J(\mathfrak{L})|^{1/3}$ that vanishes on $J(\mathfrak{L})$. We do this by linear algebra on the space of polynomials.

$$\operatorname{Poly}_D(\mathbb{R}^n) := \{\text{polynomials on } \mathbb{R}^n \text{ of degree at most } D\}.$$

A crucial fact in all our arguments is that $\operatorname{Poly}_D(\mathbb{R}^n)$ is a vector space of dimension $\sim D^n$.

Lemma 2. $\operatorname{Dim} \operatorname{Poly}_D(\mathbb{R}^n) > n^{-n} D^n$.

Proof. A basis for $\mathrm{Poly}_D(\mathbb{R}^n)$ is given by the monomials $x_1^{D_1}, ..., x_n^{D_n}$ with $\sum D_i \le D$. In particular, we can choose any degrees $0 \le D_i \le D/n$, and so the size of the basis is more than $D^n n^{-n}$. $\square$

(It's also not hard to compute the exact dimension of $\mathrm{Poly}_D(\mathbb{R}^n)$. This would lead to a slightly better constant in the joints theorem.)

Using this dimension bound, we can find polynomials of controlled degree that vanish on a given set.

Lemma 3. *If $X \subset \mathbb{R}^n$ is a finite set, then there is a non-zero polynomial Q so that*

- *Q vanishes on X*
- $\mathrm{Deg}\, Q \le n|X|^{1/n} := D$

Proof. Because of the last lemma, we can check that for $D = n|X|^{1/n}$,

$$\mathrm{Dim}\,\mathrm{Poly}_D(\mathbb{R}^n) > |X|.$$

Suppose that $X = \{x_1, x_2, ...\}$. Now we consider the evaluation map $E_X : \mathrm{Poly}_D(\mathbb{R}^n) \to \mathbb{R}^{|X|}$ defined by

$$E_X(Q) = (Q(x_1), Q(x_2), ...,).$$

The map E_X is linear, and by dimension counting, we see that the kernel of E_X is non-trivial. $\square$

By Lemma 3, we know that there is some polynomial which vanishes on the joints of $\mathfrak{L}$ and has degree at most $3|J(\mathfrak{L})|^{1/3}$. Since P has minimal degree, $\mathrm{Deg}\, P \le 3|J(\mathfrak{L})|^{1/3}$. This finishes the proof of Claim 3 and hence the proof of the main lemma. $\square$

Now we quickly explain how the main lemma implies Theorem 1. Let $J(L)$ be the maximum number of joints formed by any configuration of at most L lines. The main lemma implies that

$$J(L) \le J(L-1) + 3J(L)^{1/3}.$$

Indeed, given any configuration of L lines, one of them contains at most $3J(L)^{1/3}$ joints, and the remaining $L-1$ lines determine at most $J(L-1)$ joints.

Using this argument repeatedly, we see that

$$J(L) \le J(L-1) + 3J(L)^{1/3} \le J(L-2) + 2 \cdot 3J(L)^{1/3} \le ... \le L \cdot 3J(L)^{1/3}.$$

Rearranging gives $J(L)^{2/3} \le 3L$, and raising both sides to the power $3/2$, we get $J(L) \le 10L^{3/2}$ as desired. This finishes the proof of Theorem 1.

Comments. No one has any idea how to prove the joints theorem without mentioning polynomials. In fact, it is not easy to prove an estimate of the form $|J(\mathfrak{L})| \le CL^{1.99}$. It might be interesting for the reader to try to find their own proof of this much weaker fact.

I spent a lot of time thinking about why polynomials should play a crucial role in this question. It still seems a little mysterious to me, but I found it helpful to highlight two facts about polynomials that played a key role in this argument.

- $\mathrm{Dim}\,\mathrm{Poly}_D(\mathbb{R}^n) \sim D^n$.
- If $P \in \mathrm{Poly}_D(\mathbb{R}^n)$ vanishes at $> D$ points of a line ℓ, then $P|_\ell = 0$.

The first key fact says that there are lots of polynomials. The second key fact says that polynomials behave rather rigidly on lines. When we pick a polynomial $P \in \text{Poly}_D(\mathbb{R}^n)$, we have $\sim D^n$ degrees of freedom at our disposal, and this gives us a lot of flexibility. But then, when we consider P restricted to a line, it behaves surprisingly rigidly, with only $\sim D$ degrees of freedom. These facts show that polynomials have a special relationship with lines. The ratio between D^n and D gives us a kind of leverage which powers the proof of the theorem.

The statement of the joints theorem only involves points, lines, and planes. So it might sound reasonable at first to try to prove the joints theorem while only mentioning points, lines, and planes. The paper [GS] investigates how much one can prove in this way. It gives a list of standard properties of points, lines, and planes, and it shows that if we are only allowed to use these standard properties, then it is impossible to prove that $|J(\mathfrak{L})| \lesssim L^{1.99}$.

The proof of the joints theorem brings into play a little bit of algebraic geometry. When I started working in this area, I hadn't thought about algebraic geometry since the first semester of graduate school, and it's been interesting to go back to it with these applications in mind. My knowledge of algebraic geometry is still very basic and naive, but there is one point which puzzled me in graduate school and makes more sense now. At the beginning of my graduate class, we defined an algebraic set, and then we spent a good bit of time defining various rings of functions on an algebraic set. In order to study, for example, the zero set of a degree 3 polynomial, we began by constructing a ring involving much higher degree polynomials. It didn't seem obvious to me that this was a natural way of answering questions about a degree 3 polynomial: might we not be "reducing" these questions to harder questions? It seems to me now that this strategy is not obvious but is an important insight of algebraic geometry: we can learn a lot about a degree 3 polynomial by probing it with polynomials of all degrees. The proof of the joints theorem fits into this philosophy: in order to understand the intersection patterns of lines in $\mathbb{R}^3$, we have to probe the lines with very high degree polynomials.

2. Incidence geometry

The joints problem is easy to state, but it may look a little arbitrary. So I wanted to step back now and put it in context by talking about the field of combinatorics that it fits into, which is called incidence geometry. Incidence geometry is the field of combinatorics that studies the possible intersection patterns of lines or circles or other simple geometric shapes. In this lecture and the next lecture, I'll tell you about some of the big theorems and ideas, some hard open questions, and about the influence that this polynomial trick has had on the field.

$$\text{Let } \mathfrak{L} \text{ denote a set of } L \text{ lines in } \mathbb{R}^2.$$

An r-rich point of $\mathfrak{L}$ is a point that lies in at least r lines.

$$P_r(\mathfrak{L}) := \{x \in \mathbb{R}^2 | x \text{ lies in at least } r \text{ lines of } \mathfrak{L}\}.$$

One of the basic questions about the intersection patterns of lines in the plane is to estimate

$$P_r(L) := \max_{|\mathfrak{L}|=L} |P_r(\mathfrak{L})|.$$

Let us consider some examples. We illustrate these in Figure 2.

A. If we pick Lr^{-1} points, and we draw r lines through each point, we get a configuration of lines with Lr^{-1} r-rich points. We call this the 'stars' configuration.

B. If we pick L generic lines, then we get $\binom{L}{2} \sim L^2$ 2-rich points, but no 3-rich points.

C. A grid pattern. We can get $\sim L^2$ 3-rich points by arranging the lines in a grid, with $L/3$ vertical lines, $L/3$ horizontal lines, and $L/3$ diagonal lines at slope 1. (See the second picture in Figure 2.) If we want 4-rich points, we can add lines lines at slope -1. (See the third picture in Figure 2.) If we want 5-rich points, we can add lines at a fifth slope, such as -1/2. (See the fourth picture in Figure 2.) A calculation shows that this grid pattern yields a configuration of lines with $\sim L^2 r^{-3}$ r-rich points.

The most fundamental theorem of incidence geometry says that these examples are sharp up to a constant factor:

Theorem 4. *(Szemerédi and Trotter, 1983,* [ST]*)*

$$P_r(L) \leq C \max(Lr^{-1}, L^2 r^{-3}). \tag{ST}$$

If $r \gtrsim L^{1/2}$, then the first term dominates, and the stars example is sharp. If $r \lesssim L^{1/2}$, then the second term dominates, and the grid example is sharp.

2.1. **First upper bounds.** The first upper bounds in this problem are based on the Euclidean axiom:

$$\text{Two lines intersect in at most one point.} \tag{E}$$

We can use this lemma to bound $P_r(L)$ by a counting argument:

Lemma 5. $P_r(L) \leq \binom{L}{2}\binom{r}{2}^{-1} \sim L^2 r^{-2}$.

Proof. Suppose $\mathfrak{L}$ is a set of L lines. For each point $x \in P_r(\mathfrak{L})$, list all the *pairs* of lines in $\mathfrak{L}$ that intersect at x. For each x, we have a list of at least $\binom{r}{2}$ pairs. By the Euclidean axiom, any pair of lines intersects in at most one point, so the total number of pairs of lines in all these lists is at most $\binom{L}{2}$. Therefore, $|P_r(\mathfrak{L})|\binom{r}{2} \leq \binom{L}{2}$. □

Is this the only bound for $P_r(L)$ that follow from the Euclidean axiom? Perhaps surprisingly, there is a subtler counting argument that gives additional bounds.

Lemma 6. *If $r \geq 2L^{1/2}$, then $P_r(L) < 2Lr^{-1}$.*

For $r \geq 2L^{1/2}$, this estimate is sharp up to a constant factor, and it matches the behavior of the stars example.

Proof. Suppose that $\mathfrak{L}$ is a set of L lines. We will give a proof by contradiction, so suppose that $|P_r(\mathfrak{L})| \geq 2Lr^{-1}$. Now choose a subset $P' \subset P_r(\mathfrak{L})$ with $2Lr^{-1}$ points. Since $r \geq 2L^{1/2}$, $|P'| \leq r/2$. Each point of P' lies in r lines of $\mathfrak{L}$. But because of the Euclidean axiom, less than $r/2$ of those lines can intersect any other point of P'! Therefore, the number of lines L is bigger than $|P'|(r/2)$. But $|P'|(r/2) = (2Lr^{-1})(r/2) = L$. This contradiction shows that $|P_r(\mathfrak{L})| < 2Lr^{-1}$. □

These two lemmas give some bounds, but they don't prove Theorem 4. One may wonder if there is an even more clever way to use the Euclidean axiom to get better bounds. It turns out that the Euclidean axiom alone is not enough to prove Theorem 4.

2.2. **The main obstacle.** To understand how much we can prove with just the Euclidean axiom, we consider the following example. Let $\mathbb{F}_q$ denote a finite field with q elements. Let $\mathfrak{L}$ be a set of lines in $\mathbb{F}_q^2$. Lines in $\mathbb{F}_q^2$ obey the axiom (E), and so $|P_r(\mathfrak{L})|$ obeys the bounds in Lemmas 5 and 6. But now suppose that $\mathfrak{L}$ is the set of all lines in $\mathbb{F}_q^2$. Every point of $\mathbb{F}_q^2$ lies in $q+1$ different lines, and so $P_q(\mathfrak{L}) = \mathbb{F}_q^2$ consists of q^2 points. But $L = |\mathfrak{L}| = q^2$. The size of $P_q(\mathfrak{L})$ is sharp for Lemma 5 and much too big to be consistent with Theorem 4.

This example helps to clarify the problem and makes it more interesting. To prove Theorem 4, we have to bring into play something besides the Euclidean axiom, something which is true in $\mathbb{R}^2$ but false over finite fields. There are several nice proofs in the literature, and in some way they all use the topology of $\mathbb{R}^2$. One of the main achievements of incidence geometry is to understand how to use topology to prove combinatorial bounds. Let me show you one of the ideas for connecting topology and incidence geometry.

2.3. **The cutting method.** In addition to the L lines of $\mathfrak{L}$ let us draw D auxiliary red lines in the plane. (See Figure 3.) These D red lines cut the plane into $\sim D^2$ regions called cells. The cutting method is a divide-and-conquer argument, where we estimate the number of r-rich points in each cell and add up the results. A crucial point is that each line can only enter a small fraction of the cells.

Lemma 7. *A line can enter at most $D+1$ of the cells.*

Proof. To go from one cell to another, a line must cross one of the red lines. But a given line intersects each red line in at most one point, and so it crosses the set of red lines at most D times. □

Note that this argument uses the topology of $\mathbb{R}^2$ and doesn't make any sense over finite fields. We define the cells to be the connected components of the complement of the red lines, and so we need the idea of a connected component just to get started.

The divide-and-conquer approach works best if we can divide the problem into roughly equal pieces. Since each line enters $\sim D$ of the D^2 cells, an average cell intersects $\sim L/D$ lines of $\mathfrak{L}$. We say that the lines are (roughly) equidistributed if

$$\text{Each cell intersects } \lesssim L/D \text{ lines of } \mathfrak{L}. \qquad (EquiL)$$

If we are able to achieve equidistribution, then we can bound the number of r-rich points in each cell by $P_r(L/D)$. There could also be some r-rich points in the cell walls. We can bound the number of these points using the fact that each line intersects the cell walls in at most D places. All together, there are at most DL intersections between the lines of $\mathfrak{L}$ and the cell walls. Each r-rich point involves r intersections, and so the number of r-rich points in the cell walls is at most DL/r. So as long as we can arrange (EquiL), then we get the following bound:

$$P_r(L) \lesssim D^2 P_r(L/D) + \frac{DL}{r}. \qquad (*)$$

We can bound $P_r(L/D)$ using Lemmas 5 and 6. If we plug these bounds into $(*)$ and then optimize over D, we get the Szemerédi-Trotter bound (ST).

When I first did this computation, I thought that I understood the main idea of the proof of Theorem 4. I thought that (EquiL) wouldn't be a big deal: if we choose D red lines without thinking too much, we would probably get close to equidistribution – why would the lines of $\mathfrak{L}$ clump into only a few cells? Looking back now, I think this belief was totally wrong. Equidistribution is a very subtle and important point.

Why is equidistribution hard? For one perspective, suppose that we tried to equidistribute the points of $P_r(\mathfrak{L})$. We say that these points are (roughly) equidistributed if

$$\text{Each cell contains } \lesssim D^{-2}|P_r(\mathfrak{L})| \text{ points of } P_r(\mathfrak{L}). \qquad (EquiP)$$

In general, it is impossible to choose D red lines to equidistribute a set of points. For example, suppose that γ is a convex curve in the plane and $P_r(\mathfrak{L}) \subset \gamma$. Each red line intersects γ in at most two points, and so γ lies in at most $2D$ of the cells. So all the points of $P_r(\mathfrak{L})$ lie in only $2D$ of the $\sim D^2$ cells! (See Figure 4.)

Here is another perspective on why equidistribution is hard. A line is determined by two real parameters. So when we choose D red lines, we have $2D$ real parameters at our disposal. But (EquiL) or (EquiP) asks that each of D^2 cells has an equal share of something. So we have $2D$ real parameters and we are hoping to satisfy D^2 conditions. Roughly speaking, we have $2D$ real parameters, and we are hoping to be able to solve D^2 equations. Without some special information about these equations, this strategy sounds unlikely to work!

The paper [CEGSW] deals with the equidistribution problem using a randomization argument. They choose the D red lines to be a random subset of D lines from $\mathfrak{L}$. This set of lines has some very good properties, although it doesn't quite obey (EquiL). The paper then constructs a clever subdivision of these cells, which leads to a cell decomposition obeying (EquiL).

2.4. **Polynomial partitioning.** Using polynomials, we now sketch a different way to finish the proof of Szemerédi-Trotter, following [GK2] and [KMS]. Instead of cutting the plane into cells using D red lines, we cut it using a degree D algebraic curve: the zero set of a polynomial $P \in \mathrm{Poly}_D(\mathbb{R}^2)$. (See Figure 5.)

By the vanishing lemma, it is still true that a line can only cross $Z(P)$ in at most D points, and so it is still true that each line enters at most $D+1$ cells. So a degree D curve works just as well in the cutting method as D lines!

There are far more ways to choose a degree D curve than there are to choose D lines, and this extra flexibility allows us to achieve equidistribution. Choosing D red lines gives us only $\sim D$ parameters to play with, but choosing a polynomial $P \in \mathrm{Poly}_D(\mathbb{R}^2)$ gives us $\sim D^2$ parameters to play with, because $\mathrm{Dim}\,\mathrm{Poly}_D(\mathbb{R}^2) \sim D^2$. Equidistribution involves D^2 conditions. It turns out that we can always choose P to guarantee (EquiL) and (EquiP) (cf. [G]). Therefore, inequality $(*)$ holds (for all $D \geq 1$), and this implies the Szemerédi-Trotter theorem.

We will come back to polynomial partitioning next lecture and discuss it in more detail.

2.5. **Higher dimensions.** To end this lecture, I want to give a quick discussion of the impact of the polynomial method on incidence geometry. The main impact has to do with higher dimensions. Instead of lines in $\mathbb{R}^2$, it is natural to try to study k-planes in $\mathbb{R}^n$.

The cutting method has had some success in higher dimensions, but it also ran into a fundamental barrier. Working in $\mathbb{R}^n$, we can choose D red hyperplanes that cut space into $\sim D^n$ cells. A line can enter at most $\sim D$ of these cells, and a k-plane can enter at most $\sim D^k$ of these cells. If we have equidistribution, then we can get interesting bounds. When $k = n-1$, then we can let the D red hyperplanes be a random subset of our set of k-planes, and this leads to interesting bounds. One of the big successes of the cutting method is that it works for hyperplanes in any dimension.

But if $k < n-1$, then there is no way to choose D red hyperplanes that gets close to equidistribution. The situation is similar to trying to equidistribute points in the plane, which we discussed above. When $k < n - 1$, cutting with hyperplanes doesn't work well, and this was an important roadblock in the field for about twenty years.

The simplest case where $k < n - 1$ is $k = 1, n = 3$: lines in $\mathbb{R}^3$. This brings us back to the joints problem. The joints problem was one of the simplest open problems about the intersection patterns of lines in $\mathbb{R}^3$. It was a good test problem for exploring the regime where the codimension is bigger than 1. (We can try cutting $\mathbb{R}^3$ into cells using D red planes. If we could equidistribute the lines, the cutting method would imply that the number of joints is $\lesssim L^{3/2}$, the same bound as in Theorem 1. But in general, there is no way to choose D red planes to equidistribute a set of lines.)

The polynomial method has led to a lot of progress in the regime $k < n - 1$. The joints theorem is one good example. Also, the polynomial partitioning method works for all k and n. It doesn't solve all problems, but it has led to significant progress for all k and n, with particularly strong results about lines in $\mathbb{R}^3$. In the next lecture, we will talk more about polynomial cutting for lines in $\mathbb{R}^3$ as well as talking about some of the big open problems in incidence geometry.

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