The Schauder theory with bells and whistles

In these notes, I want to state the main theorems of the Schauder theory in a good general form, with some bells and whistles. Adding these bells and whistles puts lots of minor extra terms in the estimates, and minor extra details in the proofs. To focus on the main ideas in the proofs, I initially presented the theorems in somewhat special cases, like studying operators $Lu = \sum_{i,j} a_{ij} \partial_i \partial_j u$ without lower order terms. On the other hand, when we apply the Schauder theory to nonlinear PDE, it is genuinely convenient to have a version with lower order terms and maybe some other bells and whistles. Since we haven’t been following a textbook, I thought I would write an official statement of the theorems that you can reference. I’ll also make some small comments about the proofs.

1. The global Schauder inequality

**Theorem 1.** Suppose $0 < \alpha < 1$. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded open set with smooth boundary, and suppose

$$Lu = \sum_{i,j} a_{ij} \partial_i \partial_j u + \sum_i b_i \partial_i u + cu,$$

where $\|a, b, c\|_{C^0(\Omega)} \leq B$, and where $a_{ij}$ has eigenvalues in the range $0 < \lambda \leq a_{ij} \leq \Lambda$, and where $c \leq 0$.

Then for any $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying $Lu = f$, with $u = \phi$ on $\partial \Omega$,

$$\|u\|_{C^{2,\alpha}(\Omega)} \leq C(\lambda, \Lambda, B, n, \alpha, \Omega) \left( \|f\|_{C^{0}(\Omega)} + \|\phi\|_{C^{2,\alpha}(\partial \Omega)} \right).$$

We make a few comments about this proof. First we discuss the reason for the restriction $c \leq 0$. Modifying the proof of global Schauder from pset 2 to include a term $cu$, one naturally gets the following estimate, with an extra $\|u\|_{C^0}$ on the right-hand side:

**Theorem 2.** Suppose $0 < \alpha < 1$. Suppose $\Omega \subset \mathbb{R}^n$ is a bounded open set with smooth boundary, and suppose

$$Lu = \sum_{i,j} a_{ij} \partial_i \partial_j u + \sum_i b_i \partial_i u + cu,$$

where $\|a, b, c\|_{C^0(\Omega)} \leq B$, and where $a_{ij}$ has eigenvalues in the range $0 < \lambda \leq a_{ij} \leq \Lambda$.

Then if $u$ satisfies $Lu = f$ and $u = \phi$ on $\partial \Omega$,

$$\|u\|_{C^{2,\alpha}(\Omega)} \leq C(\lambda, \Lambda, B, n, \alpha, \Omega) \left( \|u\|_{C^0(\Omega)} + \|f\|_{C^0(\Omega)} + \|\phi\|_{C^{2,\alpha}(\partial \Omega)} \right).$$
To finish the proof, we need to use a maximum principle argument to prove an estimate of the form
\[ \|u\|_{C^0(\Omega)} \lesssim \|Lu\|_{C^0(\Omega)} + \|\phi\|_{C^0(\partial\Omega)}. \]

In this maximum principle argument, the sign of \(c\) is crucial. For example, the maximum principle fails badly for solutions of \(Lu = \triangle u + u = 0\). This happens already in one dimension, where we consider the solutions \(u(x) = h \sin x, h \in \mathbb{R}\). If \(\Omega = [0, \pi]\), then we have \(Lu = 0\) on \(\Omega\), and \(u = 0\) on \(\partial\Omega\), and yet \(\|u\|_{C^0(\Omega)} = |h|\) can be arbitrarily large.

We briefly discuss the proof of this \(C^0\) estimate using the maximum principle. If \(x_0\) is an interior maximum of \(u\), we have \(\nabla u(x_0) = 0\) and so \(\sum b_i \partial_i u(x_0) = 0\). Since \(x_0\) is a local maximum, we also know that \(\sum_{i,j} a_{ij} \partial_i \partial_j u \leq 0\). On the other hand, a priori we don’t know the sign of the term \(cu(x_0)\). We want to prove an upper bound on \(\max u\), and so we are worried about the case that \(u(x_0) \gg 0\). If \(c \leq 0\), then in this case, \(cu(x_0) \leq 0\) has the same sign as the second order term, and the argument goes through as before.

I also point out that in Theorem 1, we only assume a priori that \(u \in C^2(\Omega) \cap C^0(\bar{\Omega})\), and we conclude that \(u \in C^{2,\alpha}(\bar{\Omega})\). In class, we assumed that \(u \in C^{2,\alpha}(\bar{\Omega})\) a priori. This slight generalization just requires checking that each steps make sense for \(u \in C^2(\Omega)\). One typical thing we have to check is the formula for \(\partial_i \partial_j u\) in terms of \(\triangle u\).

**Lemma 3.** If \(u \in C_c^2(\mathbb{R}^n)\) and \(\triangle u = f\), then
\[ \partial_i \partial_j u(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} f(y) \partial_i \partial_j \Gamma(x-y) dy + \frac{1}{n} \delta_{ij} f(x). \]

The proof is similar to what we did in class, and not that interesting. I decided it’s not important to emphasize this type of detail the first time you study Schauder theory.

2. **Solving the Dirichlet problem**

The global Schauder inequality is the key to solving the Dirichlet problem, and it is possible to solve the Dirichlet problem in the same generality as we can prove the global Schauder inequality.

**Theorem 4.** Suppose \(0 < \alpha < 1\). Suppose \(\Omega \subset \mathbb{R}^n\) is a bounded open set with smooth boundary, and suppose
\[ Lu = \sum_{i,j} a_{ij} \partial_i \partial_j u + \sum_i b_i \partial_i u + cu, \]
where \( \|a, b, c\|_{C^\alpha(\Omega)} \leq B \), and where \( a_{ij} \) has eigenvalues in the range \( 0 < \lambda \leq a_{ij} \leq \Lambda \), and where \( c \leq 0 \).

Then the map

\[
    u \mapsto (Lu, u|_{\partial\Omega})
\]

is a Banach space isomorphism from \( C^{2,\alpha}(\Omega) \) to \( C^\alpha(\Omega) \oplus C^{2,\alpha}(\partial\Omega) \).

In class, we proved this theorem for \( \Omega = B_1 \). We needed to use \( B_1 \), because on \( B_1 \) the Poisson formula solves the Dirichlet problem for the Laplacian, and then we used the global Schauder estimate and the method of continuity to solve the Dirichlet problem for \( L \).

The theorem is true for any domain \( \Omega \) (any bounded open set with smooth boundary). To get the argument started, one needs to solve the Dirichlet problem for the Laplacian on the domain \( \Omega \). We state the necessary result:

**Theorem 5.** Suppose \( \Omega \subset \mathbb{R}^n \) is a bounded open set with smooth boundary. Suppose that \( \phi \in C^{2,\alpha}(\partial\Omega) \). Then there exists a function \( u \in C^2(\Omega) \cap C^0(\partial\Omega) \) so that \( \Delta u = 0 \) and \( u = \phi \) on \( \partial\Omega \).

For \( \Omega = B_1 \), the solution is given by the Poisson kernel. Also, by the global Schauder inequality, the solution \( u \) is automatically in \( C^{2,\alpha}(\bar{\Omega}) \).

If \( \Omega \) is diffeomorphic to \( B_1 \), then we can change coordinates and apply Theorem 4 with \( \Omega = B_1 \) in order to solve the Dirichlet problem on \( \Omega \), and hence prove Theorem 5. If \( \Omega \) is not diffeomorphic to \( B_1 \), then it requires another argument. We won’t do it in this course. One approach is called the Perron method of subsolutions.

A good reference for all the material discussed here is the book by Gilbarg and Trudinger. The book is very thorough, and we’ve essentially been following it in the presentation of elliptic PDE, but without being as general.