Real analysis, Further problems

Here are some further problems and questions that involve ideas from all through the class. If anyone is interested in reviewing the course later, perhaps to study for quals, it can be useful to have some problems to work on.

1. Review of the Sobolev inequality.
Suppose that $u \in C^1_{comp}(\mathbb{R}^n)$. Suppose that $p > n$. Prove that

$$\|u\|_{L^\infty} \lesssim \|u\|_p + \|\nabla u\|_p.$$

2. Here is a further problem that will delve into the proof of the Sobolev inequality. Let $\mathbb{R} \subset \mathbb{R}^2$ be the standard inclusion of the x-axis into the plane. Suppose that $u \in C^1_{comp}(\mathbb{R}^2)$. Investigate inequalities of the form

$$\|u\|_{L^p(\mathbb{R})} \lesssim \|\nabla u\|_{L^q(\mathbb{R}^2)}.$$

3. Here is a problem about the Schrodinger equation. Suppose that $u(x,t)$ solves the linear Schrodinger equation in one space dimension: $\partial_t u = i\partial_x^2 u$ with initial data $u(x,0) = f(x)$. Suppose that $\hat{f}$ is supported in $[1, 2]$ for some integer k and suppose that $\int |f|^2 = 1$. In physical terms, this is supposed to model a quantum-mechanical particle with momentum in $[1, 2]$, and $\int_A |u(x,t)|^2 dx$ is the probability that the particle lies in the region A at time t. Give an estimate for the space-time integral

(1)
$$\int_0^T \int_{-W}^{W} |u(x,t)|^2 dx dt.$$

This integral measures the expected amount of time during the time-interval $[0, T]$ that the particle spends in the region $[-W, W]$. Part of the physics intuition is that if the particle "is moving with momentum in the range $[1, 2]$", then the particle cannot stay for a long time in a small region $[-W, W]$. Can you give any rigorous mathematical justification of this intuition? What is the best estimate for (1) that you can prove in terms of the parameters T and W?

What if we consider initial data f with $\hat{f}$ supported in $[2^k, 2^{k+1}]$ instead of $[1, 2]$. Intuitively, this is supposed to model a particle moving with momentum $\sim 2^k$. How does the value of k affect your estimate for (1)?

If you want to go further, remove the restriction that the support of $\hat{f}$ lies in $[2^k, 2^{k+1}]$, and consider any initial data with $\int_\mathbb{R} |f(x)|^2 dx = 1$. For this more general class of data, what is the best estimate for (1) that you can prove in terms of T and W?

4. Here is a problem from elliptic PDE. Suppose that f and g are smooth functions on the unit ball in $\mathbb{R}^n$. Suppose that f obeys $\triangle f = f$ and g obeys $\triangle g = -g$. Suppose also that $\|f\|_{L^2(B_1)} = \|g\|_{L^2(B_1)} = 1$. What other norms of f and g can be bounded in terms of this information? For instance, does it follow that $\|f\|_{L^\infty(B_{1/2})} \lesssim 1$? What about g?

5. Consider the non-linear elliptic PDE $\triangle u = u^2 - 1$ on the unit ball $B_1 \subset \mathbb{R}^n$ with boundary data ϕ. Do you expect to be able to solve the Dirichlet problem for all boundary data $\phi \in C^{2,\alpha}$?

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What about small boundary data? Is there any other special class of boundary data that may be interesting here?

6. Given a half-space $H \subset [0,1]^2$, construct a linear operator T so that $\|Tf\|_{L^q} \leq \|f\|_{L^p}$ if and only if $(1/p, 1/q) \in H$. You might want to write T with an integral kernel, where T maps functions on $\mathbb{R}$ to functions on $\mathbb{R}$, so that

$$Tf(x) = \int_{\mathbb{R}} K(x,y)f(y)dy.$$

It might also be convenient to replace the domain and/or the range by $[0,1]$ or by $\mathbb{Z}$.