

Real analysis, Problem set 6, final project in harmonic analysis

Instead of a usual problem set, we finish the course with a more open-ended problem / project that touches on many of the topics from harmonic analysis that we have covered.

Consider the following operator defined for functions $f : \mathbb{R} \rightarrow \mathbb{C}$. We define $Tf = f * K$, where

$$K(x) = e^{ix^2} (1 + |x|^2)^{-1/4}.$$

We would like to find all the pairs (p, q) so that $\|Tf\|_{L^q(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}$. Figure out as much as you can about this problem. Can you find some (p, q) so that the estimate holds? Look at some examples! Can you find some (p, q) so that the estimate fails? Can you make a conjecture about exactly when the estimate holds? Many of the topics from our Fourier analysis unit could be helpful here: for example, interpolation, stationary phase, looking at different height scales or different frequency scales, ideas connected to Strichartz.