Real analysis, Problem set 4

In this problem set, we study the proofs of Sierpinski's estimate for the Gauss circle problem, decay estimates for PDE, and the Marcinkiewicz interpolation theorem.

1. Processing the Gauss circle problem. Do one (or more) of the following options.

Option A. The Gauss sphere problem. Let $N(R)$ be the number of integer lattice points (x, y, z) in the 3-dimensional ball of radius R, $B^3(R)$. A first approximation for $N(R)$ is the volume of $B^3(R)$. The goal is to estimate the error in this approximation:

$$E(R) := |N(R) - \mathrm{Vol}\, B^3(R)|.$$

Prove that $E(R) \leq CR^\beta$ for some $\beta < 2$. What is the best exponent β you can get? What do you guess is the truth?

Option B. Roughly circular curves in the plane. Suppose that $K \subset \mathbb{R}^2$ is a convex set with a C^2 smooth boundary whose curvature lies in the range $[1/R, 2/R]$ for some $R \geq 1$. (For comparison, a circle of radius R has curvature $1/R$ at every point.) Let $N(K)$ denote the number of integer points in K. Prove that the Sierpinski estimate extends to such a set K:

$$|N(K) - \mathrm{Area}\, K| \leq CR^{2/3}.$$

I believe that this estimate is sharp in the sense that for any $R \geq 1$, one can construct a set K as above so that $|N(K) - \mathrm{Area}\, K| \geq cR^{2/3}$. If you're interested, try to look for an example.

2. Consider the Airy equation, $\partial_t u(x, t) = \partial_x^3 u(x, t)$, for a function $u(x, t)$ with $x \in \mathbb{R}$ and $t \in \mathbb{R}$. If the initial data $u(x, 0) = f(x)$ is Schwartz, or just reasonably smooth and decaying, then the solution $u(x, t)$ can be described in terms of the Fourier transform as follows:

$$\hat{u}(\omega, t) = e^{(2\pi i\omega)^3 t} \hat{f}(\omega).$$

Prove that u obeys a decay estimate of the form

$$\|u(x, t)\|_{L^\infty_x} \leq Ct^{-\beta} \|f\|_{L^1_x},$$

and find the right exponent β.

If the computations become a little messy, you can outline the argument without writing up all details.

3. Processing interpolation. In this problem, you'll generalize the proof of interpolation that we did in class to another case. First we recall the statement of the Marcinkiewicz interpolation theorem.

Recall that an operator T is sublinear if $|T(f + g)(x)| \leq |Tf(x)| + |Tg(x)|$ and $|T(cf)(x)| = |c||Tf(x)|$. For any function F, we write

$$V_F(\lambda) := |\{x \text{ such that } |F(x)| > \lambda\}|.$$

The weak L^p "norm" is defined as

$$\|F\|_{WL^p} := \left(\sup_{0<\lambda<\infty} V_F(\lambda) \lambda^p \right)^{1/p}.$$

Here is the full Marcinkiewicz interpolation theorem.

Theorem 1. *(Marcinkiewicz) Suppose that* $1 \leq p_i, q_i \leq \infty$*.*
Suppose that T *is sublinear. Suppose that, for* $i = 0, 1$*,*

$$\|Tf\|_{WL^{q_i}} \leq A_i \|f\|_{p_i}.$$

(We make the convention that $WL^\infty = L^\infty$*.)*
Suppose that $0 < \theta < 1$*, and define* p_θ *and* q_θ *by*

$$(1/p_\theta) = \theta(1/p_1) + (1-\theta)(1/p_0),$$

$$(1/q_\theta) = \theta(1/q_1) + (1-\theta)(1/q_0).$$

Suppose that $p_0 \leq q_0$*,* $p_1 \leq q_1$*, and* $q_0 \neq q_1$*.*
Prove that

$$\|Tf\|_{L^{q_\theta}} \leq C(p_i, q_i, \theta) A_1^\theta A_0^{1-\theta} \|f\|_{L^{p_\theta}}.$$

Prove the following case of the Marcinkiewicz theorem: $p_0 = q_0 = 2$, $p_1 = 1$ and $q_1 = \infty$. You can also assume that $A_0 = A_1 = 1$ to keep the notation simpler. This case comes up pretty often: it comes up in studying the L^p estimates for the Fourier transform and in the proof of the Strichartz inequality.

You might want to prove the Marcinkiewicz theorem in full generality instead. The general case involves a little more notation but it isn't really harder than this special case.