Real analysis, Problem set 3

1. (The Poincaré Inequality.) The Poincaré inequality is a fundamental estimate about the size of a function and the size of its derivative, in the spirit of the Sobolev inequality. It is worth knowing, and proving it is a nice opportunity to digest ideas related to the Sobolev inequality. It also came up during our proof of the DeGiorgi-Nash-Moser theorem. We describe here two versions of the inequality. The first is the cleanest version, and the second is the version that we actually needed in our argument in class.

a.) The cleanest version of the Poincaré inequality is the following estimate:

Theorem 1. *If $u \in C^1([0,1]^n)$, and $\int_{[0,1]^n} u = 0$, then*

$$\int_{[0,1]^n} |u|^2 \leq C_n \int_{[0,1]^n} |\nabla u|^2.$$

b.) Here is the version that we actually used.

Theorem 2. *Suppose that B_1 is the unit ball in $\mathbb{R}^n$ and $u \in C^1(B_1)$. Suppose that*

$$|\{x \in B_1 \text{ so that } |u(x)| \leq 1\}| \geq \mu,$$

for some constant $\mu > 0$. Then prove that

$$\int_{B_1} |u|^2 \leq C(n,\mu) \left(\int_{B_1} |\nabla u|^2 + 1 \right).$$

The next several problems are designed to digest different aspects of the proof of DeGiorgi-Nash-Moser theorem.

2. Here we do some of the classical L^2-theory approach to regularity of divergence-form elliptic PDE.

Suppose that $Lu = \sum_{i,j} \partial_i (a_{ij} \partial_j u)$, where $a_{ij}(x)$ is a variable coefficient symmetric matrix with eigenvalues in the range $0 < \lambda \leq a_{ij} \leq \Lambda$, and suppose that $a \in C^1$. Suppose that $Lu = 0$ on B_1. Prove that

$$\|\partial^2 u\|_{L^2(B_{1/2})} \leq C(\lambda, \Lambda, n, \|a\|_{C^1}) \|u\|_{L^2(B_1)}.$$

Remark: With the same ideas, it is not hard to prove a similar estimate for $\|\partial^k u\|_{L^2(B_{1/2})}$. In this case, the constant depends on $\|a\|_{C^{k-1}}$.

3. (Exploring elliptic systems) Suppose that $\vec{u}$ is a vector-value function with components u_α. A divergence-form elliptic system is a system of partial differential equations of the following form:

$$(L\vec{u})_\alpha := \sum_{i,j,\beta} \partial_i \left(a_{ij\alpha\beta} \partial_j u_\beta \right) = 0 \text{ for all } \alpha.$$

Here $a_{ij\alpha\beta}(x)$ is a set of variable coefficients which is symmetric in the sense that

$$a_{ij\alpha\beta}(x) = a_{ji\beta\alpha}(x).$$

The (uniform) ellipticity of $a_{ij\alpha\beta}$ is described by giving $O < \lambda \leq \Lambda$ so that for all x,

$$\lambda |\partial \vec{u}|^2 \leq \sum_{i,j,\alpha,\beta} a_{ij\alpha\beta} \partial_i u_\alpha \partial_j u_\beta \leq \Lambda |\partial \vec{u}|^2.$$

(Here $|\partial \vec{u}|^2 = \sum_{i,\alpha} |\partial_i u_\alpha|^2$.)

Suppose that $L\vec{u} = 0$ on B_1. What aspects of the proof of DeGiorgi-Nash-Moser extend to this more general setting, and what aspects don't? How much can you say about $\vec{u}$ on $B_{1/2}$? This question is an opportunity to review the big picture ideas in the proof of DeGiorgi-Nash-Moser.

4. Global Schauder with zeroth order terms. In order to apply Schauder theory to non-linear PDE, it helps to have a version which is as general as possible. On the last pset, we proved the following theorem:

Theorem 3. *(Schauder) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary. Suppose that $Lu = \sum_{ij} a_{ij} \partial_i \partial_j u + \sum_i b_i \partial_i u$, where $0 < \lambda \leq a_{ij} \leq \Lambda$, and $\|a_{ij}\|_{C^\alpha}, \|b_i\|_{C^\alpha} \leq B$. Suppose that $u = \phi$ on $\partial\Omega$. Then:*

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leq C(n, B, \alpha, \lambda, \Lambda, \Omega) \left(\|Lu\|_{C^\alpha(\bar{\Omega})} + \|\phi\|_{C^{2,\alpha}(\bar{\Omega})} \right).$$

The operator Lu here has 2nd order terms and 1st order terms. What would happen if we add a zeroth order term? Actually the global Schauder inequality is already false for the operator $Lu = \triangle u + u$. But it holds as long as the zeroth order term has a favorable sign.

Theorem 4. *(Schauder) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary. Suppose that $Lu = \sum_{ij} a_{ij} \partial_i \partial_j u + \sum_i b_i \partial_i u + cu$, where $0 < \lambda \leq a_{ij} \leq \Lambda$ and $c \leq 0$, and $\|a_{ij}\|_{C^\alpha}, \|b_i\|_{C^\alpha}, \|c\|_{C^\alpha} \leq B$. Suppose that $u = \phi$ on $\partial\Omega$. Then:*

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} \leq C(n, B, \alpha, \lambda, \Lambda, \Omega) \left(\|Lu\|_{C^\alpha(\bar{\Omega})} + \|\phi\|_{C^{2,\alpha}(\bar{\Omega})} \right).$$

Using Theorem 3 as a black box, prove Theorem 4.

5. An application of barriers. (Thanks to Ao and Ricardo for showing me this.) The method of barriers can be used to give a nice solution to the challenge problem on the first pset.

Suppose that u is a C^2 function on the unit ball in $\mathbb{R}^n$. Suppose that $|u(x)| \leq 1$ on the ball and $|\triangle u| \leq 1$ on the ball. We will prove that $|\nabla u(0)| \leq C_n$. After a rotation, and possibly switching the sign of u, it suffices to check that $\partial_n u(0) \leq C_n$.

Let H denote the upper half-ball: $H = \{x \in \mathbb{R}^n | x_n \geq 0 \text{ and } |x| \leq 1\}$.

Let $w(x_1, ..., x_{n-1}, x_n) = u(x_1, ..., x_{n-1}, x_n) - u(x_1, ..., x_{n-1}, -x_n)$. Note that w vanishes on the plane $x_n = 0$, and that $\partial_n w(0) = 2\partial_n u(0)$. So it suffices to prove that $\partial_n w(0) \leq C_n$.

Construct an upper barrier for w on H in order to show this.