18.156 Lecture Notes

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Today, we're continuing our discussion of Sobolev inequalities from last lecture. Recall that last time, we proved the following theorem:

Theorem 1. *If $u \in C_c^1(\mathbb{R}^n)$, then*

$$\|u\|_{L^{\frac{n}{n-1}}} \leq \|\nabla u\|_{L^1}.$$

The idea of this proof was that we wrote

$$\int |u|^{\frac{n}{n-1}} \, dx_1 \cdots dx_n \leq \int u_1^{\frac{1}{n-1}} \cdots u_n^{\frac{1}{n-1}} \, dx_1 \cdots dx_n$$

where $u_i = \int |\partial_i u(x_1, \dots, x_n)| \, dx_i$ and used the Holder inequality and Fubini's theorem a lot of times. Even though this started out seeming a bit daunting, we realized that it wasn't that bad because there were a lot of paths through this mess of Holder/Fubini that led us to the right outcome. Related to what we did is the following theorem.

Theorem 2 (Gen. Loomis-Whitney)**.** *If $u_j : \mathbb{R}^n \to \mathbb{R}$ is a function of $x_1, \dots, \hat{x}_i, \dots, x_n$ where $u_j \geq 0$, then*

$$\int \prod_{j=1}^n u_j^{\frac{1}{n-1}} \leq \prod_{j=1}^n \left(\int u_j \right)^{\frac{1}{n-1}}.$$

As a sharp example of this theorem, consider

$$u_j = \prod_{i \neq j} w_i(x_i),$$

where w_i only depends on x_i and $w_i \geq 0$. Now, the left hand side of gen. Loomis-Whitney gives us

$$\int \prod_{j=1}^n u_j^{\frac{1}{n-1}} = \int \prod_{j=1}^n w_j(x_j) = \prod_{j=1}^n \int w_j$$

and the right hand side gives us

$$\prod_{j=1}^n \left(\int u_j \right)^{\frac{1}{n-1}} = \prod_{j=1}^n \prod_{i \neq j} \left(\int w_i \right)^{\frac{1}{n-1}} = \prod_{j=1} \int w_j.$$

We can use this sharp example as guidance when we're trying to figure out how to use Holder to give us our Sobolev bounds. For example, let us consider the $n = 4$ case of the above Sobolev inequality. We want to know if splitting up

$$\int \left(\int u_1^{1/3} u_2^{1/3} \cdot u_3^{1/3} u_4^{1/3} \, dx_1 \, dx_2 \right) dx_3 \, dx_4$$

is a good idea. So let us plug in the u_i from our sharp example to get

$$\int \left(\int w_2^{1/3} w_1^{1/3} \cdot (w_1 w_2)^{1/3} (w_1 w_2)^{1/3} \, dx_1 \, dx_2 \right) w_3^{?} w_4^{?} \, dx_3 \, dx_4,$$

where the question marks are some constants. And if we let $g = w_1 w_2$,

$$\int g^{1/3} g^{2/3} \leq \left(\int g \right)^{1/3} \left(\int g \right)^{2/3}$$

by Holder's inequality, where we chose the exponents to make this example work out. The idea now is that if at every step of our Holder/Fubini process, we choose exponenets to respect this example, we will probably be fine.

Question: What if we look at $\|\nabla u\|_{L^q}$ instead of $\|\nabla u\|_{L^1}$ and ask for a similar Sobolev inequality as before?

Recall that we had this issue with scaling. That is, if a Sobolev inequality held, then the exponents should hold up to scaling. Let $\eta \in C_c^\infty$ and $\eta_\lambda(x) = \eta(x/\lambda)$. Then,

$$\|\eta_\lambda\|_{L^p(\mathbb{R}^n)} = \lambda^{s_0(p,n)} \|\eta\|_{L^p(\mathbb{R}^n)} \quad \text{and} \quad \|\nabla \eta_\lambda\|_{L^q(\mathbb{R})} = \lambda^{s_1(q,n)} \|\nabla \eta\|_{L^q},$$

so we should have $s_0(p,n) = s_1(q,n)$. If we solve for these constants, we have

$$s_0(p,n) = n/p, \text{ and } s_1(q,n) = -1 + n/q.$$

Theorem 3. *If $\frac{n}{n-1} \leq p < \infty$ and the apropriate scaling holds, $u \in C_c^1(\mathbb{R}^n)$, then*

$$\|u\|_{L^p} \leq C(p,n) \|\nabla\|_{L^q}.$$

Proof. The idea here will be to convert this statement into one that we already know is true (the Sobolev inequality from last class). Let $p = \beta \cdot \frac{n}{n-1}$. Since $\beta \geq 1$, $|u|^\beta$ is C_c^1. Now, we have that

$$\begin{aligned}
\left(\int |u|^p \right)^{\frac{n-1}{n}} &= \left\| |u|^\beta \right\|_{L^{\frac{n}{n-1}}} \\
&\leq \|\nabla(|u|^\beta)\|_{L^1} \quad \text{[by original Sobolev]} \\
&\leq \beta \int |u|^{\beta-1} \cdot |\nabla u| \\
&\leq \beta \left(\int |u|^p \right)^a \left(\int |\nabla u|^q \right)^{1/q}.
\end{aligned}$$

So we have that

$$\left(\int |u|^p \right)^{\frac{n-1}{n} - a} \leq \beta \left(\int |\nabla u|^q \right)^{1/q}.$$

By scaling, we know that $\frac{n-1}{n} - a$ must equal $1/p$ and q must be the number that makes scaling hold. $\square$

The only case when no Sobolev inequality holds but the scaling equality holds is the $p = \infty$ case. Here, $p = \infty$ and $q = n$. Let us give a sketch of an example that shows why $\|u\|_{L^\infty} \lesssim \|\nabla u\|_{L^n}$ cannot hold.

Consider u radially symmetrical and $u(r) \to 0$ as $r \to \infty$. Now, we have that

$$\|u\|_{L^\infty} = u(0) = \int_0^\infty |u'(r)| \, dr \tag{1}$$

and

$$\|\nabla u\|_{L^1} = \int_0^\infty |u'(r)|^n \cdot r^{n-1} \, dr. \tag{2}$$

Our first try at a counterexample might be to take u such that $|u'(r)| = 1/r$. But this doesn't quite work since $(1) = \infty$, but $(2) = \infty$ also. But no worries. We can take something that grows slightly slower. Let us take u so that $|u'(r)| = \frac{1}{r|\log r|}$ for $0 \leq r \leq 1/e$. Then, we have that

$$(1) = \int_0^{1/e} \frac{1}{r|\log r|} \, dr = \int_1^\infty \frac{1}{s} \, ds = \infty$$

and

$$(2) = \int_0^{1/e} \frac{1}{r|\log r|^n} \, dr = \int_1^\infty \frac{1}{s^n} \, ds < \infty.$$

By taking compact cutoffs of this u, we can get that an inequality like $\|u\|_{L^\infty} \lesssim \|\nabla u\|_{L^n}$ cannot hold.

There is another kind of scaling that we could consider, and that is C^α scaling. We have then that

$$[\eta_\lambda]_{C^\alpha} = \lambda^{S_H(\alpha)} [\eta]_{C^\alpha}$$

and we may wonder whether there is a Sobolev inequality with C^α norms.

Theorem 4. *If $s_1(q, n) = s_H(\alpha)$, $0 < \alpha \leq 1$, then for all $u \in C^1(\mathbb{R}^n)$,*

$$[u]_{C^\alpha} \leq C(\alpha, n) \|\nabla u\|_{L^q}.$$

In the case when $n = 1$, this problem isn't too hard (and may have been why Holder developed the Holder inequality!). We have that

$$\begin{aligned} |u(x) - u(y)| &\le \int_x^y |\nabla u(s)| \cdot 1 \, ds \\ &\le \left(\int |\nabla u|^q \right)^{1/q} (|x-y|)^{\frac{q-1}{q}}, \end{aligned}$$

so we get that

$$[u]_{C^{\frac{q-1}{q}}} \le \|\nabla u\|_{L^q}.$$

The general case is a bit harder, and we'll get to it via the following lemma.

Lemma 5.

$$\left| u(x) - \fint_{S_x(R)} u \right| \lesssim \|\nabla u\|_{L^q} \cdot R^\alpha.$$

Proof.

$$\begin{aligned} LHS &\le \fint_{S^{n-1}} |u(x) - u(x + R\theta)| \, d\theta \\ &\le \fint_{S^{n-1}} \int_0^R |\nabla u(x + r\theta)| \, dr \, d\theta \\ &\lesssim \int_{B_x(R)} |\nabla u| \cdot r^{-(n-1)} \, dv \\ &\le \left(\int |\nabla u|^q \right)^{1/q} \left(\int_{B_R} r^{-(n-1)\frac{q}{q-1}} \right)^{\frac{q-1}{q}} \\ &= \|\nabla u\|_{L^q} \cdot R^\alpha. \end{aligned}$$

□

But this isn't quite good enough to get the bounds we want. Let us try to perturb it a little bit and show that not much changes. Let a be the midpoint of x and y, and suppose that $|x - y| = R$. Then, we claim that

$$\left| u(x) - \fint_{S_a(R)} u \right| \lesssim \|\nabla u\|_{L^q} \cdot R^\alpha.$$

In other words, moving x to a doesn't change much. To see this, we notice that

$$\begin{aligned} \left| u(x) - \fint_{S_a(R)} u \right| &\le \fint_{S_a(R)} |u(x) - u(a + R\theta)| \, d\theta \\ &\le \fint_{S^{n-1}} \left(\int_0^{2R} |\nabla u(x + r\varphi)| \, dr \right) \left| \det \frac{d\theta}{d\varphi} \right| \, d\varphi. \end{aligned}$$

But $|\frac{d\theta}{d\varphi}| \lesssim 1$ from the compactness of the sphere, so we have $|\det \frac{d\theta}{d\varphi}| \lesssim 1$ and the bounds we want hold.

So

$$\left| u(x) - \fint_{S_a(R)} u \right| \lesssim \|\nabla u\|_{L^q} \cdot R^\alpha$$

and as a result,

$$|u(x) - u(y)| \lesssim \|\nabla u\|_{L^q} \cdot R^\alpha,$$

so $[u]_{C^\alpha} \lesssim \|\nabla u\|_{L^q}$.