18.156 Lecture Notes
Lecture 7

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February 20, 2015

In Lecture 6 we developed the following continuity method for proving isomorphisms on Banach spaces:

Proposition 1. *Let X and Y be Banach spaces, let I be a connected subset of $\mathbb{R}$, and let $L_t \colon X \to Y$ be a continuous family of operators with $t \in I$. If L_{t_0} is an isomorphism for some $t_0 \in I$, and there exists $\lambda > 0$ such that $\|L_t x\|_Y \geq \lambda \|x\|_X$ for all $x \in X$ and all $t \in I$, then L_t is an isomorphism for all $t \in I$.*

We will now use α-Hölder norm estimates related to Schauder's inequality to establish an isomorphism theorem for the elliptic Dirichlet problem on discs. Let L be an elliptic operator satisfying the usual hypotheses, i.e.

$$Lu = \sum_{i,j} a_{ij} \partial_i \partial_j u$$

with $\|a_{ij}\|_{C^\alpha(B_1)} \leq \beta$ and $0 < \lambda \leq \mathrm{eig}(\{a_{ij}\}) \leq \Lambda < \infty$. Define the map $\bar{L} \colon C^{2,\alpha}(\bar{B}_1) \to C^\alpha(B_1) \times C^{2,\alpha}(\partial B_1)$ by $\bar{L}u \coloneqq (Lu, u|_{\partial B_1})$. The principal result of this lecture is:

Theorem 1. *If L obeys the usual hypotheses then $\bar{L}$ is an isomorphism.*

We may restate this result as follows:

Corollary 1. *For all $f \in C^\alpha(B_1)$ and all $\varphi \in C^{2,\alpha}(\partial B_1)$ there exists a unique $u \in C^{2,\alpha}(\bar{B}_1)$ such that $Lu = f$ on B_1 and $u|_{\partial B_1} = \varphi$.*

To establish Theorem 1, we verify that $\bar{\Delta}$ is an isomorphism, and show that $L_t \coloneqq (1-t)\Delta + tL$ satisfies the hypotheses of Proposition 1. To prove both these statements we will rely heavily on the following version of Schauder's inequality:

Theorem 2 (Global Schauder)**.** *Suppose $u \in C^{2,\alpha}(B_1)$ and L satisfies the usual hypotheses. Let $f \coloneqq Lu$ and $\varphi \coloneqq u|_{\partial B_1}$. Then*

$$\|u\|_{C^{2,\alpha}(\bar{B}_1)} \leq C(n, \alpha, \lambda, \Lambda, \beta) \left[\|f\|_{C^\alpha(B_1)} + \|\varphi\|_{C^{2,\alpha}(\partial B_1)} \right]. \tag{1}$$

The Banach spaces involved in this bound, namely $C^{2,\alpha}(B_1)$, $C^\alpha(B_1)$, and $C^{2,\alpha}(\partial B_1)$ motivate the definition of the map $\bar{L}$. Indeed, we have defined the map $\bar{L} \colon C^{2,\alpha}(\bar{B}_1) \to C^\alpha(B_1) \times C^{2,\alpha}(\partial B_1)$ because (1) is precisely the form of quantitative injectivity required to apply Proposition 1 to the family $\bar{L}_t$. We also use Theorem 2 to show:

Proposition 2. $\bar{\Delta}$ *is an isomorphism.*

Proof. From the preceding lecture, it is sufficient to show that $\bar{\Delta}$ is surjective and satisfies an injectivity estimate of the form found in Proposition 1. To prove surjectivity, fix $f \in C^\alpha(B_1)$ and $\varphi \in C^{2,\alpha}(\partial B_1)$. Extend f to $F \in C_c^\alpha(\mathbb{R}^n)$. Define $w := F * \Gamma_n$, where Γ_n is the fundamental solution to the Laplacian considered in earlier lectures. Then $w \in C^{2,\alpha}(\mathbb{R}^n)$ and $\Delta w = f$ on B_1. However, there is no reason to expect that $w|_{\partial B_1} = \varphi$. To rectify this issue, use the Poisson kernel to find $v \in C^{2,\alpha}(B_1)$ such that $\Delta v = 0$ on B_1 and $v|_{\partial B_1} = \varphi - w|_{\partial B_1}$. Set $u = v + w \in C^{2,\alpha}(B_1)$. Then $\Delta u = \Delta v + \Delta w = f$ on B_1 and $u|_{\partial B_1} = v|_{\partial B_1} + w|_{\partial B_1} = \varphi$. Hence $\bar{\Delta} u = (f, \varphi)$, so $\bar{\Delta}$ is surjective. Theorem 2 shows that

$$\|f\|_{C^\alpha(B_1)} + \|\varphi\|_{C^{2,\alpha}(\partial B_1)} \geq C(n, \alpha, 1, 1, 1)^{-1} \|u\|_{C^{2,\alpha}(\bar{B}_1)},$$

so $\left\|\bar{\Delta} u\right\| \geq \lambda \|u\|$ for all $u \in C^{2,\alpha}(\bar{B}_1)$, with $\lambda = C(n, \alpha, 1, 1, 1)^{-1} > 0$. As we showed in the previous lecture, together with surjectivity this estimate proves that $\bar{\Delta}$ is an isomorphism.

□

Proof of Theorem 1. Consider the operator L_t for $t \in [0, 1]$. Because $\|a_{ij}\|_{C^\alpha(B_1)} \leq \beta$ for all i, j,

$$\|(1-t)\delta_{ij} + t a_{ij}\|_{C^\alpha(B_1)} \leq (1-t) + t\beta \leq \beta',$$

where $\beta' := \max\{\beta, 1\}$. Similarly, we must have,

$$\operatorname{eig}(\{(1-t)\delta_{ij} + t a_{ij}\}) \subset [(1-t) + t\lambda, (1-t) + t\Lambda] \subset [\lambda', \Lambda'],$$

where $\lambda' := \min\{\lambda, 1\}$ and $\Lambda' := \max\{\Lambda, 1\}$. Hence the operators L_t obey regularity and spectral bounds which are uniform in t for $t \in [0, 1]$. Theorem 2 therefore implies that

$$\|L_t u\|_{C^\alpha(B_1)} + \|u|_{\partial B_1}\|_{C^{2,\alpha}(\partial B_1)} \geq C(n, \alpha, \lambda', \Lambda', \beta')^{-1} \|u\|_{C^{2,\alpha}(\bar{B}_1)}$$

for all $u \in C^{2,\alpha}(\bar{B}_1)$ and all $t \in [0, 1]$. By Proposition 1, this regularity combined with Proposition 2 is sufficient to establish Theorem 1. □

In summary, we used explicit formulæ involving Γ_n and the Poisson kernel to establish the surjectivity of $\bar{\Delta}$, and then use injectivity bounds furnished by the global Schauder inequality to conclude that $\bar{\Delta}$ and $\bar{L}$ are in fact isomorphisms.

It remains to verify the global Schauder inequality. We will read through the proof and fill in details for homework. The essential difference between the global and interior Schauder inequalities lies in the treatment of region boundaries. In the interior Schauder inequality proven previously, $C^{2,\alpha}$ regularity of u on a ball is controlled by C^0 regularity of Lu on a larger ball. Global Schauder replaces regularity on a larger domain with regularity on the boundary. Unsurprisingly therefore, the proof of global Schauder relies on a form of Korn's inequality which accounts for behavior near boundaries:

Theorem 3 (Boundary Korn)**.** *Let* $H := \{x \in \mathbb{R}^n;\ x_n > 0\}$ *denote the upper half space. Let* $u \in C_c^{2,\alpha}(\bar{H})$ *such that* $u = 0$ *on* ∂H*. Then* $[\partial^2 u]_{C^\alpha(H)} \leq C(,\alpha)[\Delta u]_{C^\alpha(H)}$.

As with the standard Korn inequality, the proof of Theorem 3 is divided into two parts:

1. Find a formula for $\partial_i\partial_j u$ in terms of Δu
2. Bound the integral in the formula to obtain an operator estimate on the map $\Delta u \mapsto \partial_i\partial_j u$.

To approach the first part of the proof, let $u \in C^{2,\alpha}_c(\bar{H})$, and extend Δu to $F\colon \mathbb{R}^n \to \mathbb{R}$ by setting $F(x_1, \ldots, x_n) = -\Delta u(x_1, \ldots, x_{n-1}, -x_n)$ when $x_n < 0$.

Proposition 3. *$u = F * \Gamma_n$ on $\bar{H}$.*

Proof. Let $w = F * \Gamma_n$. By the symmetry of Γ_n and the antisymmetry of F in x_n, $w = 0$ when $x_n = 0$. That is, w vanishes on ∂H. Just as in previous work, $w(x) \to 0$ as $|x| \to \infty$ and $\Delta w = F$ on H. Hence $\Delta(u - w) = 0$ on H, $u - w = 0$ on ∂H, and $u - w \to 0$ as $|x| \to \infty$. Applying the maximum principle to ever larger semidiscs, we see that $u = w$ on H. □

The same arguments from the proof of the standard Korn inequality show that

$$\partial_i\partial_j u(x) = \lim_{\varepsilon \to 0^+} \int_{|y|>\varepsilon} F(x-y)\partial_i\partial_j\Gamma_n(y)\, dy + \frac{1}{n}\delta_{ij}F(x)$$

for all $x \in H$. Define the operator $T_\varepsilon F(x) \coloneqq \int_{|y|>\varepsilon} F(x-y)\partial_i\partial_j\Gamma_n(y)\, dy$ and integral kernel $K \coloneqq \partial_i\partial_j\Gamma_n$.

On the homework we will complete the operator norm part of the proof of boundary Korn:

Proposition 4. *If $F \in C^\alpha_c(H) + C^\alpha_c(H_-)$ (but F is permitted to be discontinuous on ∂H) and $\varepsilon < \min\{x_n, \bar{x}_n\}$ with $x, \bar{x} \in H$, then*

$$|T_\varepsilon F(x) - T_\varepsilon F(\bar{x})| \leq C(n,\alpha)\, |x - \bar{x}|^\alpha \left([F]_{C^\alpha(H)} + [F]_{C^\alpha(H_-)}\right).$$

As in the proof of standard Korn, cancellation properties of K are crucial to the proof of this operator estimate. For standard Korn we used the fact that $\int_{S_r} K = 0$ for every radius r. This fact is not sufficient for boundary Korn, however, because spheres centered at x or $\bar{x}$ in H will intersect ∂H, where we have no control on F. To fix this, we note that K enjoys even stronger cancellation:

Proposition 5. *If $H_r \subset S_r$ is any hemisphere, $\int_{H_r} K = 0$.*

Proof. Γ_n is even, and hence so is its second derivative $\partial_i\partial_j\Gamma_n = K$. The substitution $y \mapsto -y$ then shows that

$$\int_{H_r} K = \frac{1}{2}\int_{S_r} K = 0.$$

□

Now to prove Proposition 4 we may divide the integral $T_\varepsilon F(x)$ into three rough regions:

1. $\varepsilon < |y| < x_n$, where K cancels on whole spheres.
2. $x_n < |y| < R$ for some large R, which is a bounded region on which K is well-behaved.
3. $|y| > R$, on which the hemisphere cancellation of K is useful.

The details of the argument are left to the homework.