18.156 Lecture Notes

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The main goal of this lecture is to prove Korn's inequality, which as we recall is as follows:

Theorem 1 (Korn's Inequality)**.** *If $u \in C^2_{comp}(\mathbb{R}^n)$, and $\Delta u = f$, then*

$$[\partial_i \partial_j u]_{C^\alpha} \leq C(n, \alpha)[\Delta u]_{C^\alpha}.$$

First, let us recall the progress that we made last time. To start, we have the following proposition allowing us to find the second partials of u.

Proposition 2. *If $u \in C^4_{comp}(\mathbb{R}^n)$, $\Delta u = f$, then*

$$\partial_i \partial_j u(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} f(y) \partial_i \partial_j \Gamma(x-y)\, dy + \frac{1}{n} \delta_{ij} f(x).$$

Since this is a bit unweildy, let us define some notation:

$$\begin{aligned} T_\epsilon f(x) &= f * K_\epsilon(x) \\ K_\epsilon(x) &= \chi_{|x|>\epsilon} \partial_i \partial_j \Gamma(x) \\ K(x) &= K_0(x) = \partial_i \partial_j \Gamma(x). \end{aligned}$$

To prove Korn's inequality, we will start by proving the following theorem.

Theorem 3. *If $f \in C^\alpha_{comp}(\mathbb{R}^n)$, then $[T_\epsilon f]_\alpha \leq C(\alpha, n)[f]_{C^\alpha}$.*

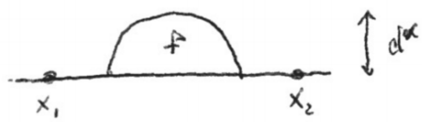
Without loss of generality, we can assume that $[f]_{C^\alpha} = 1$ and $|x_1 - x_2| = d$. Then, to prove this theorem, we want to show that

$$|T_\epsilon f(x_1) - T_\epsilon f(x_2)| \leq C(\alpha, n) d^\alpha.$$

The idea of this proof will to break up $|T_\epsilon f(x_1) - T_\epsilon f(x_2)|$ into pieces that look like behaviors that we can understand.

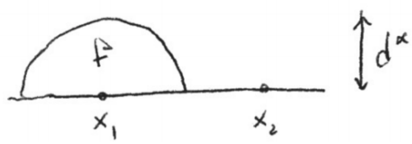
Recall that last class, we examined a few examples.

1. f supported between x_1 and x_2.



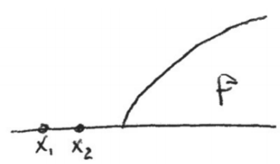
Used that $|K(x)| \lesssim |x|^{-n}$.

2. f supported over x_1.



Used that $\int_{S_r} K_\epsilon(x) = 0$ for all r, ϵ.

3. f supported on $B_{3d}(x_1)$, and $\epsilon < d$. Note that as opposed to the previous examples, $|T_\epsilon f(x_1)|$ can be $\gg d^\alpha$.



For this case, we will use the following lemma.

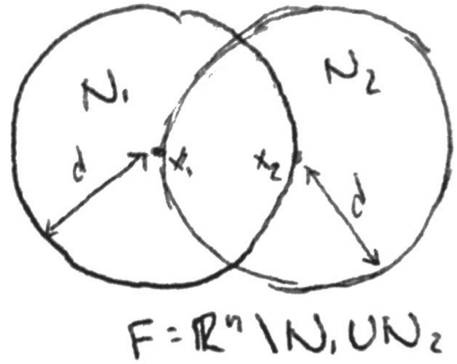
Lemma 4. *If* $|a| \leq \frac{1}{2}|b|$, *then* $|K(b) - K(b+a)| \leq |a| \cdot |b|^{-n-1}$.

With this, we have that

$$\begin{aligned}
|T_\epsilon f(x_1) - T_\epsilon f(x_2)| &= \left| \int f(y)(K(x_1 - y) - K(x_2 - y))\, dy \right| \\
&\leq \int |f(y)| d \cdot |x_1 - y|^{-n-1}\, dy \\
&\leq d \int_{|x_1 - y| > d} |x_1 - y|^\alpha |x_1 - y|^{-n-1}\, dy \\
&\lesssim d^\alpha.
\end{aligned}$$

With these examples in mind, we can now begin a proof of Theorem 3.

Proof of Theorem 3. Let us consider the main case when $\epsilon < d/10$. The picture that we should have in mind is the following.



Now we have that

$$
\begin{aligned}
|T_\epsilon f(x_1) - T_\epsilon f(x_2)| =& \left| \int f(y) K_\epsilon(x_1 - y)\, dy - \int f(y) K_\epsilon(x_2 - y)\, dy \right| \\
=& \Big| \int_{N_1} (f(y) - A) K_\epsilon(x_1 - y)\, dy - \int_{N_2} (f(y) - B) K_\epsilon(x_2 - y)\, dy \\
&\quad + \int_{N_1^c} (f(y) - C) K_\epsilon(x_1 - y)\, dy - \int_{N_2^c} (f(y) - D) K_\epsilon(x_2 - y)\, dy \Big|
\end{aligned}
$$

Let us denote the four integrands in the last expression in order by I_1, I_2, I_3, I_4. Here, the A, B, C, D may be any constants since $\int_{S_r} K_\epsilon(x) = 0$. Let us let

$$
A = f(x_1),\ B = f(x_2),\ C = D = f(a) \text{ where } a = \frac{x_1 + x_2}{2}.
$$

This way, we can leverage that $[f]_{C^\alpha}$ in our bounds. Splitting up this expression further, we have that

$$
|T_\epsilon f(x_1) - T_\epsilon f(x_2)| \leq \left| \int_{N_1} I_1 \right| + \left| \int_{N_2} I_2 \right| + \left| \int_F I_3 - I_4 \right| + \left| \int_{N_1 \setminus N_2} I_4 \right| + \left| \int_{N_2 \setminus N_1} I_3 \right|.
$$

The first two terms will behave like example 2 and the last two terms will behave like example 1. The third term will behave like example 3 and is the most interesting, so let us work through that bound.

$$
\begin{aligned}
\left| \int_F I_3 - I_4 \right| &= \left| \int_F (f(y) - f(a))(K_\epsilon(x_1 - y) - K_\epsilon(x_2 - y))\, dy \right| \\
&\leq \left| \int_F |f(y) - f(a)| \cdot d \cdot |x_1 - y|^{-n-1}\, dy \right. \\
&\leq \int_{B_{d/2}(a)^c} |a - y|^\alpha \cdot d \cdot |a - y|^{-n-1}\, dy \\
&\lesssim d^\alpha.
\end{aligned}
$$

Remark. Here we used that $\epsilon < d/10$ since the bound in the second line came from a bound on ∂K_ϵ, but K_ϵ is discontinuous. However, the $\epsilon < d/10$ means that in F we avoid this discontinuity. We also note that we didn't need to choose a to be the midpoint of x_1 and x_2. We just needed something like $|x_1 - y| \sim |a - y| \sim |x_2 - y|$ on F. $\square$

The following proposition then almost gives us Korn's inequality, except for an assumption about how many derivatives u has.

Proposition 5. *If $u \in C^4_{comp}(\mathbb{R}^n), \Delta u = f$, then $[\partial_i \partial_j u]_{C^\alpha} \lesssim [\Delta u]_{C^\alpha}$.*

Proof. Recall that for any $x_1 \neq x_2$,

$$|\partial_i \partial_j u(x_1) - \partial_i \partial_j u(x_2)| = \lim_{\epsilon \to 0^+} |T_\epsilon f(x_1) - T_\epsilon f(x_2)| + \frac{1}{n}\delta_{ij}|f(x_1) - f(x_2)|.$$

Eventually, $\epsilon < |x_1 - x_2|/10$ and we can apply theorem 3 to the first term. The second term is bounded by $[f]_{C^\alpha} \cdot |x_1 - x_2|^\alpha$. $\square$

To prove Korn's inequality, we use the **mollifier trick** to show that we only need that u has two derivatives.

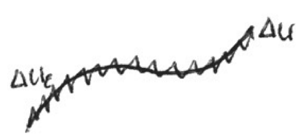
Proof of Korn's inequality. Let *varphi* $\in C_c^\infty(\mathbb{R}^n)$ be a bump function such that $\varphi \geq 0$, $\int \varphi = 1$, and define

$$\varphi_\epsilon(x) = \epsilon^{-n}\varphi(x/\epsilon),\ u_\epsilon = u * \varphi_\epsilon.$$

We have that $[\partial_i \partial_j u_\epsilon]_{C^\alpha} \lesssim [\Delta u_\epsilon]_{C^\alpha}$, and ince $u \in C_c^2$ and $u_\epsilon \in C_c^\infty$, we have that $u_\epsilon \to u$ in C^2. Now,

$$\begin{aligned} |partial_i \partial_j u(x_1) - \partial_i \partial_j u(x_2)| &= \left| \int (\Delta u(x_1 - y) - \Delta u(x_2 - y))\varphi_\epsilon(y)\, dy \right| \\ &\lesssim \liminf_{\epsilon \to 0} [\Delta u_\epsilon]_{C^\alpha}. \end{aligned}$$

Note that this isn't quite good enough, since we could have something like the following dangerous picture:



But in fact, this doesn't happen. Since $\Delta u_\epsilon = \varphi_\epsilon * \Delta u$, we have that

$$\begin{aligned}|\Delta u_\epsilon(x_1) - \Delta u_\epsilon(x_2)| &= \Big| \int (\Delta u(x_1 - y) - \Delta u(x_2 - y))\varphi_\epsilon(y)\, dy \Big| \\ &\leq [\Delta u]_{C^\alpha} |x_1 - x_2|^\alpha \int \varphi_\epsilon(y)\, dy.\end{aligned}$$

□

Our next goal will be to prove the Schauder Inequality. Recall that Korn's inequality and the first homework allowed us to prove the following lemma.

Lemma 6. *If $|a_{ij}(x) - \delta_{ij}| < \epsilon(\alpha, n)$ for all i, j, x, and $[a_{ij}]_{C^\alpha} \leq B$ on $B_1 \subset \mathbb{R}^n$, where*

$$Lu = \sum a_{ij} \partial_i \partial_j u = 0 \text{ on } B_1\, (u \in C^2(B_1)),$$

then $\|u\|_{C^{2,\alpha}}(B_{1/2}) \leq C(\alpha, n, B) \|u\|_{C^2(B_1)}$.

As a step toward proving Schauder's inequality, let us change one of the conditions in this lemma.

Proposition 7 (Baby Schauder)**.** *If $0 < \lambda \leq a_{ij} \leq \Lambda$, $[a_{ij}]_{C^\alpha(B_1)} \leq B$, $Lu = 0$ on B_1, then*

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C(\alpha, n, B, \lambda, \Lambda) \|u\|_{C^2(B_1)}.$$

Proof. First, we want to be able to replace $\delta_{ij} \leftrightarrow A_{ij}$, where $0 < \lambda \leq A_{ij} \leq \Delta$. We can do this with a change of coordinates so that $B_{1/2} \subset B_1$ becomes $E \subset 2E$, where E is an ellipse of bounded eccentricity.

Now, choose $r(\epsilon(n, \alpha), B)$ such that for $x \in B(x_0, r)$, $|a_{ij}(x_0) - a_{ij}(x)| < \epsilon(\alpha, n)$, and cover $B_{1/2}$ with such balls $B(x_i, r(i))$. Then,

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \lesssim \max_i \|u\|_{C^{2,\alpha}(B(x_i, r(i)))} \lesssim \max_i \|u\|_{C^2(B(x_i, r))} \leq \|u\|_{C^2(B_1)}.$$

□