18.156 Differential Analysis II
Lectures 31-34

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Today we start to study the Non-Linear Schrödinger Equation (NLS). Let's consider two PDEs on $\mathbb{R}^3 \times \mathbb{R}$.

FNLS (focusing NLS): $\partial_t u = i\Delta u + i|u|^2 u$

DNLS (defocusing NLS): $\partial_t u = i\Delta u - i|u|^2 u$

What we will be working towards is the following:

Theorem 1 ((Vague version))**.** *If u_0 (at $t = 0$) is smooth and decaying, then there is a solution u to DNLS on $\mathbb{R}^3 \times [0, \infty)$. For FNLS, we will show that there is a solution on $\mathbb{R}^3 \times [0, T_0)$ for some $T_0 > 0$, but not necessarily for all time.*

Remark 2. Part of our goal will be to understand why these behave so differently from each other.

Lemma 1. *If u obeys either NLS, then $\int_{\mathbb{R}^d} |u(x,t)|^2 \, dx$ is preserved.*

Proof. We have

$$\begin{aligned} \frac{d}{dt}\int |u|^2 &= \int (\partial_t u)\cdot \overline{u} + u \cdot (\partial_t \overline{u}) \\ &= 2\mathrm{Re}\int \overline{u}\cdot \partial_t u \\ &= 2\mathrm{Re}\int \overline{u}\cdot (i\Delta u \pm i|u|^2 u) \\ &= 2\mathrm{Re}\int (i\overline{u}\Delta u + i|u|^4) \end{aligned}$$

The second term $i|u|^4$ in the integrand is pure imaginary, and so contributes nothing to the real part, and using integration by parts on the first term yields that the above is equal to

$$-2\mathrm{Re}\int (i\nabla\overline{u}\cdot\nabla u),$$

and this is also zero since the integrand is pure imaginary. □

Lemma 2. *If u obeys either NLS and Ω is some domain, then*

$$\frac{d}{dt}\int_\Omega |u|^2 = \int_{\partial\Omega} N \cdot \vec{p}$$

where N is the inward pointing normal vector to the boundary and $\vec{p} = 2\mathrm{Re}(-\overline{u} i \nabla u)$.

Proof. This is the same as the last proof except for the last step when we integrate by parts, in which we pick up the extra term required. □

It is worthwhile to spend some time trying to understand how this equation behaves on an intuitive level. We consider two cases.

- Let's first consider what happens when $u_0 = \epsilon f$ for some $f \in C_c^\infty$ as $\epsilon \to 0$. In this case, we have that $|\Delta u_0| \sim \epsilon$ while $|i|u|^2 u| \sim \epsilon^3$, and so the first term dominates the evolution of u. We expect that the solution is pretty well approximated by the Schrödinger solution $e^{it\Delta} u_0$.

- Now let's consider large smooth data, say $u_0 = Af$ where $A \to \infty$. Then $|\Delta u| \sim A$ and $|i|u|^2 u| \sim A^3$, so the second term dominates, and we expect that the solution is initially well approximated by the solution of the ODE $\partial_t u = \pm i|u|^2 u$, which is just $u(x,t) = Ae^{\pm iA^2|f(x)|^2 t} f(x)$. For this u, we have $|\Delta u_t| \sim A^5 t^2 + A^3 t + A$ and $|u^3| \sim A^3$, so we expect that for t up to about order A^{-1}, this approximation should be decent, while for $t \gtrsim A^{-1}$ the $|\Delta u|$ term will kick in.

  Let's consider this small scale $u(x,t) = Ae^{\pm iA^2|f(x)|^2 t} f(x)$ and consider $\vec{p}$ as defined in the preceding proposition. We have $\nabla u = \pm iA^2 t \nabla_x(|f(x)|^2) + O(A)$, and so we find that $\vec{p}$ is approximately $\pm A^2 t|w|^2 \nabla_x(|f(x)|^2)$, where the $+$ sign is for the FNLS and the $-$ sign is for the DNLS. Qualitatively, if we draw $f(x)$ as a Gaussian or something similar, then we see that the flux is inward pointing in the focusing case and outward pointing in the defocusing case, which gives justification for these names 'focusing' and 'defocusing'.

Lemma 3. *The quantities*

$$E_D[u](t) := \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{4}|u|^4 \right)$$

$$E_F[u](t) := \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{4}|u|^4 \right)$$

are preserved for solutions to the DNLS and FNLS respectively.

Proof. We have that

$$\partial_t |\nabla u|^2 = \partial_t \left(\sum (\partial_i u)^2 \right) = 2 \sum (\partial_i \partial_t u) \cdot (\partial_i u),$$

and so integrating by parts, we find

$$\partial_t \int_{\mathbb{R}^d} |\nabla u|^2 = -2 \int_{\mathbb{R}^d} (\partial_t u) \cdot (\Delta u).$$

In fact, the left hand side is real, so we further have that this is equal to

$$\int_{\mathbb{R}^d} \mathrm{Re}(-2\partial_t u \cdot \Delta u).$$

In addition, one finds

$$\partial_t |u|^4 = \partial_t (u^2 \overline{u}^2) = 2 \left[(\partial_t u)|u|^2 \overline{u} + (\partial_t \overline{u})|u|^2 u \right].$$

If one writes this out in real and imaginary components for u, it is easy to check that this is in fact

$$\partial_t |u|^4 = \mathrm{Re}(-2i|u|^2 u \cdot \partial_t u).$$

The corresponding conservation laws now follow immediately. □

Remark 3. $E_D[u]$ will sometimes just be denoted $E[u]$, and will be called the 'energy'.

Corollary 4. If u obeys DNLS, then for all t,

$$\int_{\mathbb{R}^d} \frac{1}{2} |\nabla u_t|^2 \leq E[u](0).$$

Proof. The LHS is at most $E[u](t)$ by definition, and this is just $E[u](0)$ by the lemma. □

Remark 5. This corollary doesn't control $\|u(t)\|_{L_x^\infty}$, unfortunately, but it does in fact control $\|u(t)\|_{L_x^6}$, since this is bounded by $\|\nabla u(t)\|_{L_x^2}$ by the Sobolev inequality on $\mathbb{R}^3$, and we showed this was bounded by about $E^{1/2}$.

Remark 6. The energy we have above is Hamiltonian! First, let's recall what this means in finicte dimensions. Suppose we have $\mathbb{C}^N$ with complex coordinates $u_n = U_n + iV_n$ with U_n, V_n real coordinates. Consider a function $H\colon \mathbb{C}^N \to \mathbb{R}$, called a Hamiltonian. Then there is an associated Hamiltonian flow given by $\frac{d}{dt}U_n = \frac{\partial H}{\partial V_n}$ and $\frac{d}{dt}V_n = -\frac{\partial H}{\partial U_n}$ In particular, if there is some path tracing out the Hamiltonian flow, then we have that H is conserved along this path, because

$$\frac{d}{dt}H = \sum_n \left(\frac{\partial H}{\partial U_n}\frac{dU_n}{dt} + \frac{\partial H}{\partial V_n}\frac{dV_n}{dt} \right) = 0.$$

Now instead, the 'coordinates' themselves are indexed by $\mathbb{R}^d$, so that we have 'coordinates' $u(x) = U(x) + iV(x)$ for $u\colon \mathbb{R}^d \to \mathbb{C}$. Then one cansider the Hamiltonian function given by

$$H(u) = \int \left(\frac{1}{2}(|\nabla U|^2 + |\nabla V|^2) + h(U,V) \right) \, dx.$$

For example, we can take $h(U,V) = \frac{1}{4}|u|^4$. Hamilton's equations adopted to this case then become

$$\frac{d}{dt}U = \frac{\partial H}{\partial V} = \Delta V + \frac{\partial h}{\partial V} \qquad \frac{d}{dt}V = -\frac{\partial H}{\partial U} = -\Delta U - \frac{\partial h}{\partial U}.$$

So we have that this Hamiltonian is conserved for

$$\partial_t u = -i\Delta u + \overline{h}(u)$$

where $\overline{h}$ comes from h. If we take $h(u) = \pm\frac{1}{4}|u|^4$, we recover the NLS we are considering.

Lecture 32: 29 April 2015

Recall the set-up: we're trying to solve $\partial_t u = i\Delta u + N(u)$, in particular for $N(u) = \pm i|u|^2 u$. One thing to try is Picard iteration via the Duhamel formula for solving the inhomogeneous Schrödinger Equation, whereby we set $u_0(t) = 0$ and inductively define

$$u_{j+1}(t) = e^{it\Delta}u(0) + \int_0^t e^{i(t-\tau)\Delta}(N(u_j(\tau)))d\tau.$$

Our hope might then be that u_j converges to some actual solution u, at least on some time interval $[0,T]$. This is precisely what we will show to be the case.

Definition 7. We define the H^s norm of a function by

$$\|f\|^2_{H^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} |\widehat{f}(w)|^2(1+|w|)^{2s}dw.$$

Note that this is easily seen to be a norm if we just write it as $\|\widehat{f}(w)(1+|w|)^s\|_2$.

Lemma 4. *For s an integer, $\|f\|_{H^s} \sim \|f\|_2 + \|\partial^s f\|_2 \sim \sum_{j=0}^s \|\partial^j f\|_2$.*

Proof. This is just a sketch - the details are left for the reader. Write out $(1+|w|)^{2s}$ using binomial coefficients. Then note that $|\widehat{\partial^k f}| \sim |w^k \widehat{f}|$ up to a constant. It is then easy to get bounds to prove the lemma. □

Lemma 5. $\|e^{it\Delta}f\|_{H^s} = \|f\|_{H^s}$

Proof. Immediate from the fact that $\widehat{e^{it\Delta}f} = e^{iC|w|^2 t}\widehat{f}$ where $C = (2\pi i)^2$ is real. □

Proposition 8. *If $u(0) \in H^s$ for $s \geq 4$ an integer, then on $[0,T]$, we have $u_j(t) \to u(t)$ in H^s uniformly in t for some $u(t)$ which solves the NLS (so $u \in C^2$, for example), and where $T = T(\|u(0)\|_{H^s})$ is a monotonically decreasing function of the H^s norm of the starting condition.*

This proof comes pretty easily from the following lemma:

Lemma 6. *$\|u_j(t)\|_{H^s} \leq 2\|u(0)\|_{H^s}$ for all j and all $0 \leq t \leq T$ (this $T > 0$ will be defined in the proof).*

In order to prove this, we will need a certain 'product lemma,' which occurs further as a result of the following:

Lemma 7 (Sobolev Inequality)**.** *If $s > d/2$ (not necessarily an integer), then*

$$\|f\|_{L^\infty(\mathbb{R}^d)} \leq C(s)\|f\|_{H^s(\mathbb{R}^d)}.$$

Proof. We have that this is a simple application of Cauchy-Schwarz:

$$\begin{aligned}
\|f\|_\infty &\leq \|\widehat{f}\|_1 \\
&= \|\widehat{f}(1+|w|)^s(1+|w|)^{-s}\|_1 \\
&\leq \|\widehat{f}(1+|w|)^s\|_2\|(1+|w|)^{-s}\|_2 \\
&= \|f\|_{H^s}\|(1+|w|)^{-s}\|_2
\end{aligned}$$

and if $s > d/2$, then $\|(1+|w|)^{-s}\|_2 < \infty$. □

Lemma 8 (Product Lemma)**.** *If $s > d$ is an integer, then*

$$\|fg\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{H^s}.$$

Proof. We have

$$\begin{aligned}
\|fg\|_{H^s}^2 &\sim \sum_{|\alpha|\leq s} \|\partial^\alpha(fg)\|_2^2 \\
&= \sum_{|I|+|J|\leq s} \int_{\mathbb{R}^d} |\partial^I f|^2|\partial^J g|^2
\end{aligned}$$

Now either $|I| \leq s/2$ or $|J| \leq s/2$ for each term in the sum. If $|I| \leq s/2$, then we have that for any $0 < \epsilon < 1/2$,

$$\|\partial^I f\|_\infty \lesssim \|\partial^I f\|_{H^{d/2+\epsilon}} \lesssim \|\partial^I f\|_{H^{s/2}} \lesssim \|f\|_{H^s}$$

where the first inequality is by Sobolev, the second is since $s > d$, and the last is since $|I| \leq s/2$. Hence, for such I, we have that the terms have bound

$$\int |\partial^I f|^2|\partial^J g|^2 \leq \|\partial^I f\|_\infty^2 \int |\partial^J g|^2 \lesssim \|f\|_{H^s}^2\|g\|_{H^s}^2.$$

We have similar bounds for the $|J| \leq s/2$ terms. Finally, the number of terms depends only on s and d, so we are done. □

Proof of Lemma 6. We proceed by induction. We clearly have the result for $u_0 = 0$.

$$\begin{aligned}
\|u_{j+1}(t)\|_{H^s} &\leq \|e^{it\Delta}u(0)\|_{H^s} + \|\int_0^t e^{i(t-\tau)\Delta}(\pm i|u_j|^2u_j(\tau))\,d\tau\|_{H^s} \\
&\leq \|u(0)\|_{H^s} + \int_0^t \||u_j|^2u_j\|_{H^s}\,d\tau \\
&\leq \|u(0)\|_{H^s} + C\int_0^t \|u_j\|_{H^s}^3 d\tau \\
&\leq \|u(0)\|_{H^s} + 8Ct\|u(0)\|_{H^s}^3
\end{aligned}$$

where the first inequality is just the triangle inequality, the second inequality is the Minkowski inequality, the third inequality is from the product lemma where C depends only on s, d, and the last inequality is by the inductive hypothesis. It is now clear that the induction holds if we choose $T = (8C)^{-1}\|u(0)\|_{H^s}^{-2}$. □

Remark 9. We will later prove a Hard Product Lemma (next lecture) which will allow us to apply the same product lemma but for $s > d/2$ and not necessarily an integer. This could be used to give a more general version of Lemma 6.

Lemma 9. *For some $T = T(\|u(0)\|_{H^s})$ monotonically decreasing, and $s \geq 4$ (in particular s is an integer greater than $d = 3$), we have*

$$\max_{0\leq t\leq T} \|u_{j+1}(t) - u_j(t)\|_{H^s} \leq \frac{1}{2} \max_{0\leq t\leq T} \|u_j(t) - u_{j-1}(t)\|_{H^s}.$$

Proof. We have

$$\begin{aligned}\|u_{j+1}(t) - u_j(t)\|_{H^s} &= \| \int_0^t e^{i(t-\tau)\Delta}((\pm i|u_j|^2 u_j) - (\pm i|u_{j-1}|^2 u_{j-1}))\ d\tau\|_{H^s} \\ &\leq \int_0^t \||u_j|^2 u_j - |u_{j-1}|^2 u_{j-1}\|_{H^s}\ d\tau\end{aligned}$$

Now, we would like to use the product lemma on this integrand. We see in particular that

$$|u_j|^2 u_j - |u_{j-1}|^2 u_{j-1} = |u_j|^2 (u_j - u_{j-1}) + u_j u_{j-1}\overline{(u_j - u_{j-1})} + (u_j - u_{j-1})|u_{j-1}|^2$$

and so applying the triangle inequality and the Product Lemma yields now that

$$\||u_j|^2 u_j - |u_{j-1}|^2 u_{j-1}\|_{H^s} \leq (\|u_j\|^2 + \|u_j\|\|u_{j-1}\| + \|u_{j-1}\|^2)\|u_j - u_{j-1}\| \leq \frac{3}{2}(\|u_j\|^2 + \|u_{j-1}\|^2)\|u_j - u_{j-1}\|.$$

Hence, if we choose t small enough so that Lemma 6 applies, then we have that

$$\|u_{j+1}(t) - u_j(t)\|_{H^s} \leq 12t\|u(0)\|_{H^s}^2 \max_{0\leq\tau\leq t} \|u_j - u_{j-1}\|_{H^s}.$$

Therefore, the T needed is just $T = \min\{(8C)^{-1}, 1/24\} \cdot \|u(0)\|_{H^s}^{-2}$ where C is as in the proof of Lemma 6. □

Proof of Proposition 8. From the preceding lemma, we have that $u_j(t) \to u(t)$ in H^s uniformly in $t \in [0, T]$. Now recall the Picard Iteration formula

$$u_{j+1}(t) = e^{it\Delta}u(0) + \int e^{i(t-\tau)\Delta}(\pm i|u_j|^2 u_j)\ d\tau.$$

The left hand side converges uniformly in H^s to $u(t)$ as we just said. The first term on the right hand side is just the same. Meanwhile, the integral term also converges uniformly in H^s by the Product Lemma and Lemma 5 for the integrand and the Minkowski inequality for H^s to take care of the integral. Hence, we have

$$u(t) = e^{it\Delta}u(0) + \int e^{i(t-\tau)\Delta}(\pm i|u|^2 u)\ d\tau,$$

and this just matches the Duhamel formula for the solution to $\partial_t u = i\Delta u \pm i|u|^2 u$. □

So we have succeeded in proving existence for at least some small time interval, but we are interested in proving the global existence of the DNLS. One might wonder how close we are. Well, one strategy might be to find the solution for $[0, T]$ where T depends on $\|u(0)\|$, and then to see how far we can continue from $\|u(T)\|$ as inital condition. But this won't get us very far! That is, we know $T \sim \|u(0)\|^{-2}$, and that $\|u(T)\| \leq 2\|u(0)\|$, so we only know we can continue another $T/4$, then another $T/4^2$, and so on. This only gets us up to $T/3$. (Of course, we didn't expect this to work because if it did, then it would also work for the FNLS, which we claimed has no global solutions.)

Lecture 33: 1 May 2015

Consider the PDE $\partial_t u = i(\Delta u \pm u^3)$, which looks similar to our NLS situation, but is slightly changed. Note that if $\Delta u \pm u^3 = 0$, then we get a steady state solution. However, with the minus sign, which is almost the defocusing case, there is a maximum principle, so any $u(0)$ that decays to zero at ∞ and has $\Delta u - u^3 = 0$ must be just 0, and so this 'D'NLS has no steady state solutions. Meanwhile, the 'F'NLS considered here will have solutions to $\Delta u + u^3 = 0$ decaying to zero at ∞, and so will have steady state solutions.

We remark that without this non-linear part, Strichartz implies automatically that every solution to the Schrödinger equation will eventually decay, so the example above does fundamentally show how an NLS can be qualitatively quite different from the linear Schrödinger Equation.

33.1 The Hard Product Lemma and Persistence of Regularity

Changing focus back to the problem at hand, we had shown last time how to use the easy product lemma to obtain a solution to the FNLS and DNLS for a time interval which was about the size of $T \sim \|u(0)\|_{H^s}^{-2}$. We would like to do a bit better for the DNLS. Recall in particular that we had $\|u(T)\|_{H^s} \leq 2\|u(0)\|_{H^s}$, but this was not good enough to extend for all time. Our first main task is to find a better version of the product lemma, which will be used to obtain better bounds on $\|u(T)\|_{H^s}$ and will in turn show the existence of a global solution for the DNLS (though the whole story will only be completed next lecture). Let us first state this Hard Product Lemma.

Lemma 10 (Hard Product Lemma). *For all $s \geq 0$,*

$$\|fg\|_{H^s} \lesssim \|f\|_\infty \|g\|_{H^s} + \|f\|_{H^s}\|g\|_\infty.$$

We will prove it later, but for now, let us look at why this might be hard. For s an integer, we have that the H^s norm squared is equivalent to a sum of L^2 norms squared for up to s derivatives, so we can look at $s = 0, 1, 2$ to start.

- For $s = 0$, the inequality is trivial because $\|fg\|_2 \leq \|f\|_\infty \|g\|_2$
- For $s = 1$, the inequality is still relatively easy. Namely,

  $$\|fg\|_{H^1}^s = \|fg\|_2^2 + \|(\nabla f)g\|_2^2 + \|f(\nabla g)\|_2^2$$

  and each term is boundable in the same way as the $s = 0$ case.
- For $s = 2$, we could try the same computation, but now there is a $\|(\nabla f)(\nabla g)\|_2^2$ term which is tricky. So we need another tool.

Corollary 10. For $s > d/2$, $\|fg\|_{H^s} \lesssim \|f\|_{H^s}\|g\|_{H^s}$.

Proof. This is just the Hard Product Lemma combined with the Sobolev Inequality from last lecture to bound the L^∞ terms. □

Remark 11. For $s < d/2$, the statement of the corollary is false. One can take $f = g$ with compact support and scale so that the support lies inside a small enough ball.

We continue to delay the proof of the Hard Product Lemma for now, and turn to why it is useful. We can actually generalize our DNLS and FNLS examples to consider instead $\partial_t u = i\Delta u \pm i|u|^{p-1}u$ for p an odd integer. We will denote this by NLS_p. Then we have the following corollary of the Hard Product Lemma, which finally gives a bound on $\|u(T)\|_{H^s}$ which will be more tractable to show global existence of a solution to DNLS (by which I mean DNLS_3). In particular, the estimate now interplays with $\|u(t)\|_{L_x^\infty}$ for times $0 \leq t \leq T$. We will see next lecture how to use this.

Corollary 12 (Persistence of regularity). If u obeys NLS_p on $[0, T]$, then

$$\|u(T)\|_{H^s} \leq \|u(0)\|_{H^s} \exp\left(C(s)\int_0^T \|u(t)\|_{L_x^\infty}^{p-1} dt\right)$$

Proof. By the Duhamel formula, we have

$$u(T) = e^{iT\Delta}u(0) + \int_0^T e^{i(T-t)\Delta}(\pm i|u|^{p-1}u(t))dt.$$

Hence, taking the H^s norm, we have

$$\begin{aligned}\|u(T)\|_{H^s} &= \|e^{-iT\Delta}u(T)\|_{H^s} \\ &= \|u(0) + \int_0^T e^{-it\Delta}(\pm i|u|^{p-1}u)dt\|_{H^s}\end{aligned}$$

We therefore find that

$$\begin{aligned}\frac{d}{dT}\|u(T)\|_{H^s} &\leq \|e^{-iT\Delta}(\pm i|u(T)|^{p-1}u(T))\|_{H^s} \\ &\leq C(s)\|u(T)\|_{L^\infty_x}^{p-1}\|u(T)\|_{H^s}\end{aligned}$$

where the last inequality is by the Hard Product Lemma. In particular, dividing out the $\|u(T)\|_{H^s}$ term yields

$$\frac{d}{dT}\log\|u(T)\|_{H^s} \leq C_s\|u(T)\|_{L^\infty_x}^{p-1}$$

and integrating this yields the desired result. □

33.2 Proof of the Hard Product Lemma

We turn our attention back to the Hard Product Lemma. We note that the H^s norm is all about integrating the Fourier transform of the function against a $(1+|w|)^{-2s}$ term. It might therefore make sense to want to split our function into its frequency components, and this goes back to Littlewood-Paley which we covered a number of lectures ago. We modify some of the notation slightly just for our convenience.

- For $k \geq 1$, set $A_k := \{w\colon 2^{k-1} \leq |w| \leq 2^{k+1}\}$
- Fix a partition of unity $\{\psi_k\}_{k\geq 0}$ subordinate to the open cover of the frequency domain by $B(2)$ and the A_k, i.e. such that $\mathrm{supp}\psi_0 \subseteq B(2)$ and $\mathrm{supp}\psi_k \subseteq A_k$ for $k \geq 1$.
- Define $P_k f := (\psi_k \widehat{f})^\vee$.

Lemma 11. $\|P_k f\|_{H^s} \sim 2^{ks}\|P_k f\|_{L^2}$

Proof. First suppose $k \geq 1$. We have

$$\begin{aligned}\|P_k f\|_{H^s}^2 &= \int |\widehat{P_k f}|^2(1+|w|)^{2s}dw \\ &= \int_{A_k} |\widehat{P_k f}|^2(1+|w|)^{2s}dw\end{aligned}$$

where the second equality is because $\widehat{P_k f} = \psi_k\widehat{f}$ is supported on A_k. Then $2^{k-1} \leq |w| \leq 2^{k+1}$ on this region, and since $k \geq 1$, we therefore have $2^{k-1} \leq 1+|w| \leq 2^{k+2}$, so we immediately find that up to constants (which depend on s),

$$\|P_k f\|_{H^s}^2 \sim 2^{2ks}\|\widehat{P_k f}\|_2^2 = 2^{2ks}\|P_k f\|_2^2,$$

where we have used Plancherel's theorem to obtain the last inequality.

In the case where $k = 0$, the same trick works. □

Lemma 12. *If $k \leq \ell$, then*

$$\|P_k f \cdot P_\ell g\|_{H^s} \lesssim \|P_k f\|_{L^\infty} \cdot \|P_\ell g\|_{H^s}$$

Proof. The square of the left hand side is simply $\int |\widehat{P_k f} * \widehat{P_\ell g}|^2(1+|w|)^{2s}dw$, and we see that the integrand has support on $A_k + A_\ell \subseteq B(10\cdot 2^\ell)$ (where $A_0 = B(2)$ to include the $k=0$ case). Hence, we see that $1+|w| \leq C2^\ell$ and we get that

$$\|P_k f \cdot P_\ell g\|_{H^s}^2 \lesssim 2^{2\ell s}\|\widehat{P_k f\cdot P_\ell g}\|_2^2.$$

Applying Plancherel and Hölder, we thus obtain

$$\begin{aligned}\|P_k f \cdot P_\ell g\|_{H^s}^2 &\lesssim 2^{2\ell s}\|P_k f\cdot P_\ell g\|_2^2\\ &\leq 2^{2\ell s}\|P_k f\|_\infty^2\|P_\ell g\|_2^2\end{aligned}$$

but the previous lemma thus allows us to replace $2^{2\ell s}\|P_\ell g\|_2^2$ with $\|P_\ell g\|_{H^s}^2$, and we obtain the desired result. ☐

This last result tells us that the Littlewood-Paley parts themselves satisfy the correct inequality, and this is good indication that we're on our way. Let us first build up our arsenal of weapons (or array of tools, if you're a pacifist) a little more before we finally proof the Hard Product Lemma. Really, we want to eventually understand bounds on f and not on the pieces $P_k f$, so let's do that.

Lemma 13. $\|f\|_{H^s} \sim \sum_k \|P_k f\|_{H^s}$

Proof.

$$\begin{aligned}\|f\|_{H^s}^2 &= \int |\widehat{f}|^2(1+|w|)^{2s}\, dw\\ &= \int \left|\sum_k \psi_k \widehat{f}\right|^2 (1+|w|)^{2s}\, dw\\ &\sim \int \sum_k |\psi_k\widehat{f}|^2(1+|w|)^{2s}\, dw\\ &= \sum_k \|P_k f\|_{H^s}^2\end{aligned}$$

where the asymptotic comes from the fact that each ψ_k is positive and the supports of the ψ_k form a locally finite cover of $\mathbb{R}^d$ with constant $O(1)$ (i.e. a point is in at most N of the A_k for some fixed constant N). ☐

Lemma 14. $\|P_k f\|_{L^q} \lesssim \|f\|_{L^q}$ *for all* $1 \leq q \leq \infty$*, where the constant is independent of* q *and* k*.*

Proof. We have $P_k f = f * \psi_k^\vee$, and so we have $\|P_k f\|_{L^q} \leq \|f\|_{L^q}\|\psi_k^\vee\|_{L^1}$, so it suffices to bound $\|\psi_k^\vee\|_{L^1}$. We sketch the details. We can set up so that $|\psi_k| \leq 1$, $|\partial\psi_k| \lesssim 2^{-k}$, $|\partial^2\psi_k| \lesssim 2^{-2k}$, and so on. Then $|\psi_k^\vee(x)| \lesssim 2^{kd}$ on $B(2^{-k})$ and rapidly decays for $|x| > 2^{-k}$, and so $\|\psi_k^\vee\|_{L^1} \lesssim 1$. ☐

For convenience in notation, we will define $P_{\leq k} f := \sum_{\ell\leq k} P_k f$ and similarly let $\psi_{\leq k} = \sum_{\ell \leq k}\psi_\ell$ so that $P_{\leq k} f = (\psi_{\leq k}\widehat{f})^\vee$. We have that the previous lemma applies also to $P_{\leq k}$. Finally, with this notation, we come to the proof we've been delaying:

Proof of the Hard Product Lemma. We write

$$\|fg\|_{H^s}^2 = \int_{\mathbb{R}^d}\left|\sum_{k,\ell}\widehat{P_k f} * \widehat{P_\ell g}\right|^2 (1+|w|)^{2s}\, dw.$$

Now we can break this up into three sums to bound. The first is when $k \leq \ell - 4$, the second when $\ell \leq k-4$, and the last when $|k-\ell| \leq 3$. Let us try to bound these parts of the sum.

- First consider the indices with $k \leq \ell - 4$. We have

$$\begin{aligned}
\int \left| \sum_{k \leq \ell-4} \widehat{P_k f} * \widehat{P_\ell g} \right|^2 (1+|w|)^{2s}\, dw &\lesssim \sum_\ell \int \left| \widehat{P_{\leq \ell-4} f} * \widehat{P_\ell g} \right|^2 (1+|w|^2)^s\, dw \\
&\lesssim \sum_\ell 2^{2\ell s} \|P_{\leq \ell-4} f P_\ell g\|_2^2 \\
&\lesssim \sum_\ell \|P_{\leq \ell-4} f\|_{L^\infty}^2 \cdot 2^{2\ell s} \|P_\ell g\|_2^2 \\
&\lesssim \|f\|_\infty^2 \sum_\ell \|P_\ell g\|_{H^s}^2 \\
&\sim \|f\|_\infty^2 \|g\|_{H^s}^2
\end{aligned}$$

  where the $2^{2\ell s}$ comin the second inequality came from noting that the support of that integral was contained in a ball of radius $|w| \sim 2^\ell$.

- The $\ell \leq k - 4$ case is symmetric.

- Now consider just the case of $|k - \ell| \leq 3$. In this case, we have that the sets $B_{k,\ell} = A_k + A_\ell$ form a locally finite cover in this restricted range, so we can just exchange our sums and absolute values at will since $\widehat{P_k f} * \widehat{P_\ell g}$ will be supported on $B_{k,\ell}$. In particular, we find:

$$\begin{aligned}
\int \left| \sum_{|k-\ell| \leq 3} \widehat{P_k f} * \widehat{P_\ell g} \right|^2 (1+|w|)^{2s}\, dw &\sim \sum_{|k-\ell|\leq 3} \int \left| \widehat{P_k f} * \widehat{P_\ell g} \right|^2 (1+|w|)^{2s}\, dw \\
&\lesssim \sum_{|k-\ell|\leq 3} \int \left| \widehat{P_k f} * \widehat{P_\ell g} \right|^2 2^{2\ell s}\, dw \\
&\lesssim \sum_{|k-\ell|\leq 3} 2^{2\ell s} \|(P_k f)(P_\ell g)\|_2^2 \\
&\lesssim \sum_\ell 2^{2\ell s} \left[\sum_{k=\ell-3}^{\ell} \|P_k f\|_\infty^2 \|P_\ell g\|_2^2 + \sum_{k=\ell}^{\ell+3} \|P_k f\|_2^2 \|P_\ell g\|_\infty^2 \right] \\
&\lesssim \sum_\ell \left[\|f\|_\infty^2 \|P_\ell g\|_{H^s}^2 + \sum_{k=\ell}^{\ell+3} \|P_k f\|_{H^s}^2 \|g\|_\infty^2 \right] \\
&\lesssim \|f\|_\infty^2 \|g\|_{H^s}^2 + \|f\|_{H^s}^2 \|g\|_\infty^2
\end{aligned}$$

Combining these three pieces yields the result. (We've used a lot of the preceding tools and lemmas in this proof implicitly, and the active reader should attempt to understand each inequality in this proof.) □

Lecture 34: 4 May 2015 (Last Lecture)

Let us list what we already know. Recall that NLS_p is just the equation $\partial_t u = i\Delta u \pm i|u|^{p-1}u$, where p is an odd integer, and we are thinking of the equation on $\mathbb{R}^3 \times \mathbb{R}$. We have the following facts (some of which we only proved for $p = 3$, but which are essentially the same otherwise).

- If $u(0) \in H^s$ for $s > 3/2$, then there is a solution for time $T = c\|u(0)\|_{H^s}^{-2}$.
- We had conservation of energy $E = \int \frac{1}{2}|\nabla u|^2 \pm \frac{1}{p+1}|u|^{p+1}$ and mass $M = \int |u|^2$. For the defocusing case, this implies that $\|u(t)\|_{H^1} \lesssim (M+E)^{1/2}$.
- There was a persistence of regularity, so that $\|u(T)\|_{H^s} \leq \|u(0)\|_{H^s} \exp\Big(C(s) \cdot \int_0^T \|u(t)\|_{L_x^\infty}^{p-1}\, dt\Big)$.

34.1 Finishing the Proof - the Key Estimate and Bootstrapping

We turn now to the last piece of the puzzle. With the persistence of regularity in place, all we need is the following:

Proposition 13 (Key Estimate)**.** *If u obeys* NLS$_3$ *(either version) on $[0,T]$ where $T \le T_0(\|u(0)\|_{H^1})$ then*

$$\int_0^T \|u\|^2_{L^\infty_x} \le 1.$$

Corollary 14. If u solves DNLS$_3$ on $[0,T]$ for any T, then

$$\|u(T)\|_{H^s} \le \|u(0)\|_{H^s} \exp(CT/T_0).$$

Proof. Combine the Key Estimate with the fact that there is abound for $\|u(t)\|_{H^1}$ for this defocusing case. □

Corollary 15. We have global solutions to DNLS$_3$.

So it remains to prove the Key Estimate! This is not actually very obvious and will rely on a new idea. But let us first do a warm-up lemma.

Lemma 15 (Warm-Up 'Linear Version')**.** *There is some $\alpha > 0$ such that $\int_0^T \|e^{it\Delta}u(0)\|^2_{L^\infty_x}\, dt \lesssim T^\alpha \|u(0)\|^2_{H^1}$.*

Proof. Recall that Strichartz tells us that for $\sigma = 2 \cdot \frac{d+2}{d} = \frac{10}{3}$, we have the estimate

$$\|e^{it\Delta}u(0)\|_{L^\sigma_{x,t}} \le K\|u(0)\|_{L^2}$$

for some constant K. In particular, this is bounded by $K\|u(0)\|_{H^1}$. Then note that ∇_x commutes with both Δ and ∂_t, so $e^{it\Delta}(\nabla_x u(0)) = \nabla_x e^{it\Delta}u(0)$. Hence, we find therefore that

$$\|\nabla_x e^{it\Delta}u(0)\|_{L^\sigma_{x,t}} \le K\|\nabla_x u(0)\|_{L^2} \le K\|u(0)\|_{H^1}.$$

Now we can use a variant of the Sobolev inequality we proved two lectures ago, we obtain:

$$\|e^{it\Delta}u(0)\|_{L^\infty_x} \lesssim \|e^{it\Delta}u(0)\|_{L^\sigma_x} + \|\nabla e^{it\Delta}u(0)\|_{L^\sigma_x}.$$

Finally, putting it all together,

$$\begin{aligned}
\int_0^T \|e^{it\Delta}u(0)\|^2_{L^\infty_x}\, dt &\lesssim \int_0^T \left(\|e^{it\Delta}u(0)\|_{L^\sigma_x} + \|\nabla_x e^{it\Delta}u(0)\|_{L^\infty_x}\right)^2 \\
&\lesssim T^{1-\frac{2}{\sigma}} \left(\int_0^T \left(\|e^{it\Delta}u(0)\|_{L^\sigma_x} + \|\nabla_x e^{it\Delta}u(0)\|_{L^\infty_x}\right)^\sigma\right)^{2/\sigma} \\
&\lesssim T^{1-\frac{2}{\sigma}} \|u(0)\|^2_{H^1}
\end{aligned}$$

□

In that warm-up, what was really important? At the end, all we needed was that $\|e^{it\Delta}u(0)\|_{L^\sigma_{x,t}[0,T]}$ and that $\|\nabla_x e^{it\Delta}u(0)\|_{L^\sigma_x}$ were bounded by some constant times $\|u(0)\|_{H^1}$. We really need to do this up to some time T_0. The trick for doing this is called bootstrapping. The lemma is provided below, and following the lemma, we will show how this actually proves the Key Estimate.

Lemma 16 (Bootstrap Lemma)**.** *Suppose u obeys* NLS$_3$ *on $[0,T]$ with $T \le T_0 = T_0(\|u(0)\|_{H^1})$ (which will be discovered in the proof) and*

$$\|u(t)\|_{H^1} \le 4\|u(0)\|_{H^1}$$

$$\|u\|_{L^\sigma_{x,t}([0,T])} \le 4K\|u(0)\|_{H^1}$$

$$\|\nabla_x u\|_{L^\sigma_x} \le 4K\|u(0)\|_{H^1}.$$

The constant K is the same one as in the proof of the previous lemma. Let us call these conditions A,B,C, respectively. Under these hypotheses, we have that the same estimates hold on $[0,T]$ but where the constants in the inequalities are changed from 4 to 2. We will call these changed hypotheses A',B',C'.

Proof. Proving C' from A,B,C is the hardest of the three, so we'll just do that. We have $\partial_t u = i\Delta u \pm i|u|^2 \cdot u$. Then we find by applying ∇_x that

$$\partial_t \nabla_x u = i\Delta \nabla_x u + F(u)$$

where $|F| \lesssim |u|^2 |\nabla_x u|$. So by the Strichartz inequality for the inhomogeneous Schrödinger equation, with σ' the dual exponent to σ, we have

$$\|\nabla_x u\|_{L^\sigma_{x,t}} \leq K\|\nabla_x u(0)\|_{L^2} + C\|F\|_{L^{\sigma'}_{x,t}}.$$

So in order to get C', it suffices to show $\|F\|_{L^{\sigma'}_{x,t}([0,T])} \lesssim T^\alpha \|u(0)\|^3_{H^1}$ because then there will be a natural $T_0(\|u(0)\|_{H^1})$ such that for $T \leq T_0$ we will have $C\|F\|_{L^{\sigma'}_{x,t}} \leq K\|u(0)\|_{H^1}$. Since $|F| \lesssim |u|^2 \cdot |\nabla u|$, it suffices bound $\||u|^2 \cdot |\nabla u|\|_{L^{\sigma'}_{x,t}}$ in similar fashion. So let's do this.

Before reading this paragraph, I wish to point out that σ and 6 are not the same, but in my handwritten notes they look the same. I have tried to be careful about which one is the correct one, but I very well might have transcribed incorrectly. Of course, the dyslexic might still find that σ and 6 look quite similar in form, and so it might still be confusing. Now combining the Sobolev inequality with Condition A gives that $\|u(t)\|_{L^6_x} \lesssim \|u(t)\|_{H^1} \leq 4\|u(0)\|_{H^1}$. Meanwhile, we also have that since $\sigma' = 10/7$, we can use Hölder to find that

$$\||u|^2 \cdot |\nabla u|\|_{L^{\sigma'}_x} \lesssim \|u\|^2_{L^6_x} \|\nabla u\|_{L^q_x}$$

with $q = 30/11$, which we note is less than $\sigma = 10/3$. So combining the two inequalities in this short paragraph, we find that

$$\||u|^2 \cdot |\nabla u|\|_{L^{\sigma'}_x} \lesssim \|u(0)\|^2_{H^1} \|\nabla u\|_{L^q_x}.$$

Therefore, we find

$$\begin{aligned} \||u|^2 \cdot |\nabla u|\|^{\sigma'}_{L^{\sigma'}_{x,t}([0,T])} &= \int_0^T \||u|^2 \cdot |\nabla u|\|^{\sigma'}_{L^{\sigma'}_x} \\ &\lesssim \|u(0)\|^{2\sigma'}_{H^1} \int_0^T \|\nabla u\|^{\sigma'}_{L^q} \end{aligned}$$

Now note that $2 \leq q \leq \sigma$ since $q = 30/11$ and $\sigma = 10/3$. In particular, there is some constant α (different from the $1 - 2/\sigma$ one in the previous lemma) such that $\|\nabla u\|_q \lesssim \|\nabla u\|^\alpha_\sigma \|\nabla u\|^{1-\alpha}_2$. Substituting this back in above and using Condition C, we now get a bound of the form

$$\|u(0)\|^{(3-\alpha)\sigma'}_{H^1} \int_0^T \|\nabla u\|^{\sigma'\alpha}_\sigma.$$

Now we can use Hölder to get

$$\int_0^T \|\nabla u\|^{\sigma'\alpha}_{L^\sigma} \lesssim T^\beta \left(\int_0^T \|\nabla u\|^\sigma_\sigma\right)^{\alpha\sigma'/\sigma}$$

where $\beta = 1 - \frac{\alpha\sigma'}{\sigma}$ (which is OK since $\sigma' \leq \sigma$ and $\alpha < 1$. So therefore we have now found

$$\||u|^2 \cdot |\nabla u|\|^{\sigma'}_{L^{\sigma'}_{x,t}([0,T])} \lesssim \|u(0)\|^{(3-\alpha)\sigma'}_{H^1} T^\beta \left(\int_0^T \|\nabla u\|^\sigma_\sigma\right)^{\alpha\sigma'/\sigma},$$

and using Condition C on this last term, we find once and for all that

$$\||u|^2 \cdot |\nabla u|\|^{\sigma'}_{L^{\sigma'}_{x,t}([0,T])} \lesssim \|u(0)\|^{3\sigma'}_{H^1} T^\beta,$$

which completes the proof. □

So how do we now finally prove the Key Estimate? Well, A',B',C' imply the Key Estimate on the interval $[0,T]$ in which they are satisfied. So suppose now that u satisfies NLS_p on $[0,T]$ with $T \le T_0(\|u(0)\|_{H^1})$. Then let $S \subseteq [0,T]$ be the set on which A,B,C are satisfied, and S_0 the component of this in which 0 is contained ($u(0)$ trivially satisfies these with 4 replaced with 1). Note that if $u(0)$ is nonzero, then in fact the ratios $\|u(t)\|_{H^1}/\|u(0)\|_{H^1}$ and similar ratios corresponding to B and C will be continuous, so S_0 is a closed interval of the form $[0,T']$ for some $T' \le T$. Suppose by way of contradiction that $T' \ne T$. Then on S_0, the Bootstrapping Lemma tells us that not only A,B,C are satisfied, but also A',B',C', and by the continuity of the ratios in question, S_0 extends past T', which contradicts the fact that $S_0 = [0,T']$.

Remark 16. In fact, the Key Estimate then gives existence of the solution to the DNLS for all time, and so the above reasoning actually proves that the solution to the DNLS satisfies A,B,C, and hence A',B',C', for all $t \in [0,T_0]$.

Remark 17. It might be worth reiterating what goes wrong for the FNLS case, since it might not be clear. The point is that T_0 depends on $\|u(0)\|_{H^1}$, and so for DNLS, the important thing was that $\|u(t)\|_{H^1}$ is always bounded above by $(M+E)^{1/2}$. This allowed us to combine what we knew about how long we could extend our solution based on $\|u(0)\|_{H^2}$ to keep extending by enough that we will eventually cover all time. For FNLS, we have no such bound, and so we might not be able to keep extending by the entirety of that amount! It is worth writing out a proof of Corollary 14 to see how this works!

34.2 Final Remarks

Question 1. If u obeys NLS_p on $[0,T]$ with $T \le T_0(\|u(0)\|_{H^1})$ for some function T_0 then does $\int_0^T \|u\|_{L^\infty_x}^{p-1}\, dt \le 1$?

There's also a linear version:

Question 2. Is it true that

$$\int_0^T \|e^{it\Delta}u(0)\|_{L^\infty_x}^{p-1}\, dt \lesssim T^\alpha \|u(0)\|_{H^1}$$

for some $\alpha \ge 0$?

This is a bit more approachable to start. Note that we can use scaling, i.e. by forming $u(x/\lambda, t/\lambda^2)$, and so if it is true for this linear version, then $\alpha = 1 - \frac{1}{4}(p-1)$, and so we see that for $p \ge 7$ (remember p is an odd integer) we would have $\alpha < 0$, which is nonsense. For $p < 5$, we have $\alpha > 0$, and this is in fact true. The tricky case of $p = 5$ with $\alpha = 0$ also turns out to be true. In fact, bootstrapping will always work here.

How about what is known about global solutions for DNLS_p? In the '70s and '80s, this was solved for $p < 5$. The $p = 5$ case is in fact very delicate, and in particular, for the first question, the answer is NO in general. However, global solutions DO exist! This was solved throughout the '90s and '00s by Bowgain and Colliander-Keel-Staffilani-Takaoka-Tao.

For $p > 5$, this is still an open question, and so is the mathematics of the future (and extrapolating from previous data points, should be solved in the '10s and '20s).