18.156 Lecture Notes
Lecture 3

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In Lecture 2 we mentioned that

$$\left\|\partial^2 u\right\|_{C^0} \not\lesssim \|\Delta u\|_{C^0} \quad \text{and} \quad \left\|\partial^2 u\right\|_{C^1} \not\lesssim \|\Delta u\|_{C^1} \tag{1}$$

for $u \in C_c^3(\mathbb{R}^n)$. However, Korn showed that

$$[\partial^2 u]_\alpha \lesssim [\Delta u]_\alpha$$

for any $a \in (0,1)$. We will spend the next few lectures proving results involving the α-Hölder norms, culminating in a proof of the Schauder inequality. In the process we will also see why the bounds in (1) fail. The α-Hölder norms provide an intermediate measure of smoothness between C^0 and C^1, and offer valuable control on solutions to $\Delta u = f$ and $\Delta u = 0$ with boundary conditions.

To approach Korn's inequality, we will use an expression for $\partial^2 u$ in terms of Δu. This formula is derived from physical potential theory. The "gravitational" field in $\mathbb{R}^n$ is given by

$$F_n(x) = c_n \frac{x}{|x|^n}$$

for some constant $c_n > 0$ and $x \neq 0$. As we shall see, it is most convenient to choose $c_n = \frac{1}{|S^{n-1}(1)|}$ to simplify divergence formulæ for F_n. In fact F_n is generated by the potential

$$\Gamma_n(x) = \begin{cases} c_n' \, |x|^{-n+2} & \text{if } n \geq 3 \\ c_2' \log |x| & \text{if } n = 2 \end{cases}$$

for appropriate constants $c_n' > 0$. That is, $F_n = \nabla \Gamma_n$. Is is easy to calculate that $|\nabla \Gamma_n| \sim |x|^{-n+1}$ and $\left|\nabla^2 \Gamma_n\right| \sim |x|^{-n}$ for large $|x|$. Furthermore, $\operatorname{div} F_n = 0$ at $x \neq 0$, so $\Delta \Gamma_n = 0$ for $x \neq 0$. We note in passing that $\operatorname{div} F_n = 0$ may be derived without computation from the symmetry of F_n and the fact that

$$\int_{S^{n-1}(r)} F_n \cdot \hat{n} = 1 \tag{2}$$

is independent of r.

For the remainder of the lecture, let $\Omega \subset \mathbb{R}^n$ be an open bounded region with smooth boundary $\partial\Omega$. From $\operatorname{div} F_n = 0$ for $x \neq 0$ and (2), we may easily verify that

$$\int_\Omega F_n \cdot \hat{n} = \begin{cases} 1 & \text{if } 0 \in \Omega \\ 0 & \text{if } 0 \notin \Omega. \end{cases} \tag{3}$$

This reflects the physical (and distribution theoretic) interpretation that $\operatorname{div} F_n = \Delta\Gamma_n = \delta_0$. We now verify that convolution against Γ_n yields a solution to Poisson's equation.

Proposition 1. *If $f \in C_c^2(\mathbb{R}^n)$ and $u := \Gamma_n * f$, then $u \in C^2(\mathbb{R}^n)$ and $\Delta u = f$.*

Proof. By definition,

$$u(x) = \int_\Omega f(y)\Gamma_n(x-y)\,dy = \int_\Omega \Gamma_n(y)f(x-y)\,dy.$$

These integral expressions are well-defined because $f \in C_c^0(\mathbb{R}^n)$ and $\Gamma_n \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Also, standard dominated convergence arguments show that we may bring first derivatives under the integral sign:

$$\partial_j u = \int_\omega f(y)\partial_j\Gamma_n(x-y)\,dy = \int_\Omega \Gamma(y)\partial_j f(x-y)\,dy.$$

Again these expressions are well-defined because $f \in C_c^1$ and $\partial_j\Gamma_n \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Differentiating further, we have

$$\partial_i\partial_j u = \int_\Omega \Gamma(y)\partial_i\partial_j f(x-y)\,dy.$$

Note that we may not form a parallel expression with $\partial_i\partial_j\Gamma_n$, because $\partial_i\partial_j\Gamma_n \notin L^1_{\mathrm{loc}}(\mathbb{R}^n)$. By continuity, it is sufficient to verify that

$$\int_\Omega \Delta u = \int_\Omega f$$

for all regions Ω satisfying the previously stated conditions. By the divergence theorem,

$$\int_\Omega \Delta u = \int_{\partial\Omega} \nabla u \cdot \hat{n} = \int_\Omega \left(\int_{\mathbb{R}^n} f(y)\nabla\Gamma_n(x-y)\,dy \right) \cdot \hat{n}\,dA(x).$$

We use Fubini to interchange the order of integration:

$$\int_\Omega \Delta u = \int_{\mathbb{R}^n} f(y) \left(\int_\Omega \nabla\Gamma_n(x-y) \cdot \hat{n}\,dA(x) \right) dy.$$

Now (3) implies that

$$\int_\Omega \Delta u = \int_{\mathbb{R}^n} f(y)\chi_\Omega(y)\,dy = \int_\Omega f.$$

□

Having proven a solution to the equation $\Delta u = f$, the question of uniqueness naturally arises. Could other expressions for u also solve Laplace's equation? We establish uniqueness in the case that u is compactly supported:

Proposition 2. *If $u \in C_c^4(\mathbb{R}^n)$ and $f := \Delta u$, then $u = \Gamma_n * f$.*

Proof. Let $w := \Gamma_n * f$, so $\Delta(u - w) = 0$. The maximum principle for harmonic functions shows that $\max_{B_R} |u - w| = \max_{\partial B_R} |u - w|$. Hence to verify that $u = w$ it is sufficient to show that

$$\lim_{R\to\infty} \max_{\partial B_R} |u - w| = 0.$$

Since u is compactly supported, this is equivalent to showing that

$$\lim_{R\to\infty} \max_{\partial B_R} |w| = 0.$$

This is simple when $n \geq 3$. After all, if $\operatorname{supp} u \subset B_{R_0}$ and $|x| = R \geq 2R_0$, we have

$$|w(x)| = \left| \int_{\mathbb{R}^n} f(y)\Gamma_n(x - y)\, dy \right| \leq \|f\|_{L^1(\mathbb{R}^n)} \sup_{|z|\geq R/2} |\Gamma_n(z)| \to 0$$

as $R \to \infty$. This estimate fails when $n = 2$, because Γ_2 does not decay as $|x| \to \infty$. We therefore deploy a more careful analysis, relying on the fact that f is the Laplacian of a compactly supported function. In particular,

$$\int_{\mathbb{R}^2} f(y)\, dy = \int_{B_{R_0}} \Delta u(y)\, dy = \int_{S_{R_0}} \nabla u(y) \cdot \hat{n}\, dA(y) = 0.$$

Hence when $|x| = R \geq 2R_0$,

$$\begin{aligned} |w(x)| &= \left| \int_{B_{R_0}} f(y)\Gamma_2(x)\, dy + \int_{B_{R_0}} f(y)[\Gamma_2(x - y) - \Gamma_2(x)]\, dy \right| \\ &= \left| \int_{B_{R_0}} f(y)[\Gamma_2(x - y) - \Gamma_2(x)]\, dy \right| \\ &\leq \int_{B_{R_0}} |f(y)|\, |y| \max_{[x,x-y]} |\nabla\Gamma_2|\ dy \\ &\leq R_0\, \|f\|_{L^1(\mathbb{R}^2)} \sup_{|z|\geq R/2} |\nabla\Gamma_2(z)| \to 0 \end{aligned}$$

as $R \to \infty$.

□

Korn's inequality bounds the regularity of $\partial^2 u$ in terms of Δu for compactly supported functions. We therefore wish to adapt the expressions in Proposition 1 to derive formulæ for the second partials of u. However, as noted in the proof of Proposition 1, this goal is complicated by the fact that $\partial_i\partial_j\Gamma_n \notin L^1_{\text{loc}}(\mathbb{R}^n)$. Hence we may not directly write $\partial_i\partial_j u = (\Delta u) * \partial_i\partial_j\Gamma_n$. We might hope that the integral defining the convolution converges conditionally, i.e. that

$$\partial_i\partial_j u = \lim_{\varepsilon\to 0^+} \int_{|x-y|>\varepsilon} f(y)\partial_i\partial_j\Gamma_n(x - y)\, dy \tag{4}$$

for $u \in C^4_c(\mathbb{R}^n)$. However, this equation is patently false if we recall that $\Delta\Gamma_n(z) = 0$ for $z \neq 0$. If we use (4) with $i = j$ and sum over $1 \leq i \leq n$, we find $\Delta u = 0$, regardless of the choice of u. Hence we need to account somehow for the effect of the singularity of Γ_n on derivatives of the convolution $(\Delta u) * \Gamma_n$. As it turns out,

(4) is *almost* correct:

Proposition 3. *If $f \in C_c^2(\mathbb{R}^n)$ and $u := f * \Gamma_n$, then*

$$\partial_i \partial_j u(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} f(y) \partial_i \partial_j \Gamma_n(x-y)\, dy + \frac{1}{n} \delta_{ij} f(x). \tag{5}$$

Proof. As noted in the proof of Proposition 1, we may certainly write

$$\partial_i \partial_j u(x) = \partial_i \int_{\mathbb{R}^n} f(y) \partial_j \Gamma_n(x-y)\, dy = \partial_i \int_{\mathbb{R}^n} f(x-y) \partial_j \Gamma_n(y)\, dy = \int_{\mathbb{R}^n} \partial_i f(x-y) \partial_j \Gamma_n(y)\, dy.$$

The game of switching the convolution arguments between y and $x-y$ is necessary because the derivatives ∂_i and ∂_j act on x, not y. Because $\partial_j \Gamma_n$ is locally integrable, we have

$$\begin{aligned}
\partial_i \partial_j u(x) &= \lim_{\varepsilon \to 0^+} \int_{|y|>\varepsilon} \partial_i f(x-y) \partial_j \Gamma_n(y)\, dy \\
&= \lim_{\varepsilon \to 0^+} \partial_i \int_{|y|>\varepsilon} f(x-y) \partial_j \Gamma_n(y)\, dy \\
&= \lim_{\varepsilon \to 0^+} \partial_i \int_{|x-y|>\varepsilon} f(y) \partial_j \Gamma_n(x-y)\, dy.
\end{aligned}$$

We wish to once again move the derivative ∂_i inside the integral, but the region of integration now depends on x. Accounting for this:

$$\partial_i \int_{|x-y|>\varepsilon} f(y) \partial_j \Gamma_n(x-y)\, dy = \int_{|x-y|>\varepsilon} f(y) \partial_i \partial_j \Gamma_n(x-y)\, dy + \int_{|x-y|=\varepsilon} (\hat{x}_i \cdot \hat{n}) f(y) \partial_j \Gamma_n(x-y)\, dy.$$

Hence

$$\partial_i \partial_j u(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} f(y) \partial_i \partial_j \Gamma_n(x-y)\, dy + \lim_{\varepsilon \to 0^+} \int_{|x-y|=\varepsilon} (\hat{x}_i \cdot \hat{n}) f(y) \partial_j \Gamma_n(x-y)\, dy.$$

To complete the proof, we need to compute the second integral on the right hand side. As $\varepsilon \to 0^+$, we note that $\hat{x}_i \cdot \hat{n} = \mathcal{O}(1)$, $f(y) = f(x) + \mathcal{O}(\varepsilon)$, and $\partial_j \Gamma_n(x-y) = \mathcal{O}(\varepsilon^{-n+1})$. The region of integration is a sphere of volume $\mathcal{O}(\varepsilon^{n-1})$. We therefore see that we may replace $f(y)$ by $f(x)$ in the integral to achieve the same limit. That is:

$$\begin{aligned}
\lim_{\varepsilon \to 0^+} \int_{|x-y|=\varepsilon} (\hat{x}_i \cdot \hat{n}) f(y) \partial_j \Gamma_n(x-y)\, dy &= f(x) \lim_{\varepsilon \to 0^+} \int_{|x-y|=\varepsilon} (\hat{x}_i \cdot \hat{n}) \partial_j \Gamma_n(x-y)\, dy \\
&= f(x) \lim_{\varepsilon \to 0^+} \int_{S^{n-1}(\varepsilon)} c_n \frac{z_i z_j}{|z|^{n+1}}\, dA(z) \\
&= \frac{1}{n} \delta_{ij} f(x).
\end{aligned}$$

$\square$