18.156 Lecture Notes

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Today's class will be split up into a discussion of the last problem set and then a continuation of our discussion of Calderon-Zygmund.

1 Pset 4, Problem 3

We're going to start today with a discussion of problem 3 on the previous homework assignment (problem set 4). A lot of people tried to prove that

$$V_{Tf_k}(2^\ell) \lesssim |S_k(f)|^{q_\theta/p_\theta} 2^{kq_\theta} 2^{-\ell q_\theta} 2^{-\epsilon|k-\ell|}.$$

Unfortunately, this isn't quite true. Instead, let

$$A := |S_k(f)|^{q_\theta/p_\theta} 2^{kq_\theta} 2^{-\ell q_\theta}.$$

We get two bounds from our two $\|Tf_k\|_{q_i} \lesssim \|f_k\|_{p_i}$ bounds, and we should let $\bar{\ell}$ be the value of ℓ where the two things that we get from these bounds are equal to each other, and use $\bar{\ell}$ instead of k. Then,

$$V_{Tf_k}(2^\ell) \lesssim A 2^{-\epsilon|\ell-\bar{\ell}|}$$

and when $|\ell - \bar{\ell}|$ is big, we have a gain. We note here that $\bar{\ell}$ depends on both K and $|S_k(f)|$. We want when

$$|S_k(f)|^{q_1/p_1} 2^{kq_1} 2^{-\ell q_1} = |S_k(f)|^{q_0/p_0} 2^{kq_0} 2^{-\ell q_0}.$$

We can solve this for $\bar{\ell}$ if $q_0 \neq q_1$. If $q_1/q_0 \neq p_1/p_0$, then $|S_k(f)|$ matters. For the special case when $q_1 = \infty$, then $V_{Tf_k}(2^\ell) = 0$ if $2^\ell \gg \ldots$ and $\bar{\ell}$ is the biggest ℓ consistent with the L^∞ bound. Now,

$$\begin{aligned}
\|Tf_k\|_{q_\theta}^{p_\theta} &\sim \sum_\ell V_{Tf_k}(2^\ell) 2^{\ell q_\theta} \\
&\leq \sum_\ell |S_k(f)|^{q_\theta/p_\theta} 2^{kq_\theta} 2^{-\epsilon|\ell-\bar{\ell}|} \sim \|f_k\|_{p_\theta}^{q_\theta}.
\end{aligned}$$

Now, we want to try to combine all of the Tf_k. We have two extreme cases. In the first case, we could have that $k \mapsto \bar{\ell}(k)$ is injective, in which case we can use weights. In the econd case, $f_k \neq 0 \leftrightarrow k = 1, \ldots, N$ and $\bar{\ell}(k) = 0$ for all $k = 1, \ldots, N$. Then,

$$\|Tf\|_{q_\theta} \lesssim \sum_k \|Tf_k\|_{q_\theta} \lesssim \sum_k |S_k(f)|^{1/p_\theta} 2^k,$$

and we want this $\lesssim (\sum_k |S_k(f)|2^{kp_\theta})^{1/p_\theta}$. But having $\bar{\ell}(k) = 0$ for all $k = 1, \ldots, N$ gives a formula for $|S_k(f)|$ and $|S_k(f)|^{1/p_\theta} 2^k$ gives a geometric series. We get then that

$$2^{\bar{\ell}} = (2^k)^\alpha |S_k(f)|^\beta.$$

2 Calderon-Zygmund

Let's go back to the Calderon-Zygmund decomposition lemma. Let us state it again here:

Lemma 1. *For $f \in C_c^0$, $\lambda > 0$, we can decompose $f = b + s$, the sum of a balanced part and a small part, such that $\|b\|_1 + \|s\|_1 \lesssim \|f\|_1$ and $\|s\|_\infty \leq \lambda, b = \sum b_j$ where b_j are balanced for λ supported on disjoint Q_j and*

$$\fint_{Q_j} b_j \lesssim \fint_{Q_j} f \lesssim \lambda.$$

Proof. We're going to use a Calderon-Zygmund iterated stopping time algorithm to construct Q_j and b_j. Start with a cubical grid in $\mathbb{R}^d$ of side length s large and $\fint_Q |f| < \lambda$ in each cube.

[Call this point in the algorithm (A).] Now, consider each Q.

(i) If $\fint_Q |f| < \lambda$, subdivide Q into 2^d equally sized cubes and repeat this step (A) with each of the subdivided cubes.

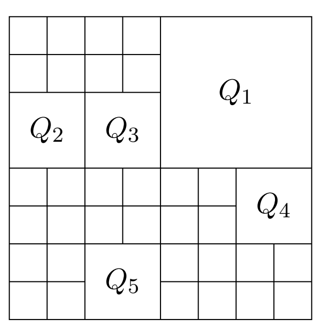
(ii) If $\fint_Q |f| \geq \lambda$, add Q to the list of balanced cubes, call it Q_j, and let

$$b_j = f \cdot \chi_{Q_j} - \fint_{Q_j} f.$$

Do not go back to (A) with this cube.

The output of the algorithm is $\{Q_j\}$ and a function b_j for each Q_j. Then, let

$$b = \sum b_j, \quad s = f - b.$$



We can make some observations now. First,

$$\lambda \leq \fint_{Q_j} |f| < 2^d \lambda.$$

We also have some bound for s. If $x \notin \bigcup Q_j$, then

$$|s(x)| = |f(x)| \leq \lambda.$$

If $x \in Q_j$, then

$$|s(x)| = |f(x) - b_j(x)| = \left| \fint_{Q_j} f \right| \leq \fint_{Q_j} |f| \leq 2^d \lambda,$$

and so we have that

$$\int_{\mathbb{R}^d \setminus \bigcup Q_j} |s| = \int_{\mathbb{R}^d \setminus \bigcup Q_j} |f| \leq \|f\|_{L^1}.$$

From this, we get that

$$\int_{\bigcup Q_j} |s| \leq \int_{\bigcup Q_j} |f| \leq \|f\|_{L^1},$$

so $\|s\|_1 \leq \|f\|_1$. We also have bounds for the b_j:

$$\fint_{Q_j} |b_j| = \fint_{Q_j} \left| f - \fint_{Q_j} f \right| \leq 2 \int_{Q_j} |f|$$

and

$$\int_{Q_j} b_j = \int_{Q_j} f - \int_{Q_j} f = 0.$$

□

This lemma then helps us conclude part II of the proof of Calderon-Zygmund, since $V_{Tf}(2\lambda) \leq V_{Ts}(\lambda) + V_{Tb}(\lambda)$. By the L^2 bound $V_{Tf}(\lambda \lesssim \|f\|_1 \lambda^{-1}$. We also have that

$$\begin{aligned}
V_{Tb}(\lambda) &\leq \left| \bigcup_j 2Q_j \right| + \lambda^{-1} \int_{\mathbb{R}^d \setminus \bigcup 2Q_j} |Tb| \\
&\lesssim |\bigcup_j Q_j| + \sum_j \lambda^{-1} \int_{\mathbb{R}^d \setminus 2Q_j} |Tb_j| \\
&\lesssim \|f\|_1 \lambda^{-1} + \lambda^{-1} \sum_j \|b_j\|_1 \\
&\lesssim \lambda^{-1}(\|s\|_1 + \|b\|_1) \\
&\lesssim \lambda^{-1} \|f\|_1.
\end{aligned}$$

Part III: Interpolation. Since we have a weak L^1 bound and a strong L^2 bound, we can use Marcinkiewicz interpolation to get the bound $\|Tf\|_p \lesssim \|f\|_p$ for $1 < p \leq 2$.