18.156 Lecture Notes

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trans. Jane Wang

We're going to start off today with some applications of interpolation. Then, we'll move on to discussing the Calderon-Zygmund Theorem.

1 Applications of Interpolation

1.1 Hardy-Littlewood Maximal Function

Our first application will be about the Hardy-Littlewood maximal function. Let $f : \mathbb{R}^d \to \mathbb{C}$. Then the maximal function of f is defined to be

$$Mf(x) := \sup_{r>0} \fint_{B(x,r)} |f|.$$

For example, if $f = \chi_{B(0,1)}$, then $Mf(x) \sim (1+|x|)^{-d}$ and $\|Mf\|_{L^p} < \infty$ if and only if $p > 1$.

Theorem 1. *$\|Mf\|_p \leq C(d,p)\|f\|_{L^p}$ for all d, p with $1 < p \leq \infty$.*

Notice that our above example shows why we could not expect an inequality like this to hold if $p = 1$. Also, we remark that we always have an inequality $\|Mf\|_\infty \leq \|f\|_\infty$. Now, to apply the Marcinkiewicz interpolation theorem to get the above bound for all $1 < p \leq \infty$, it suffices to find a weak L^1 bound. This is the content of the next lemma.

Lemma 2 (Weak L^1)**.** $V_{Mf}(\lambda) \cdot \lambda \lesssim \|f\|_{L^1}$.

Proof. Suppose $Mf(x) > \lambda$. Then, there would exist an $r > 0$ for which $\fint_{B(x,r)} |f| > \lambda$. Let us call such a $B(x,r)$ a **λ-dense ball**. Then,

$$\{x : Mf(x) > \lambda\} \subset \bigcup \{\lambda\text{-dense balls}\}.$$

Now recall the Vitali Covering Lemma: if $\{B_j\}_{j\in J}$ is a finite list of balls, then there exists $I \subset J$ such that $\{B_j\}_{j\in I}$ are disjoint and

$$\bigcup_{i\in I} 3B_i \supseteq \bigcup_{j\in J} B_j.$$

However, we can't use this this because our collection of λ-dense balls is not finite. However, we recall the infinite version of the Vitali convering lemma that says that if the supremum of the radius of our balls is finite, then the Vitali covering lemma holds with 3 replaced by 5 and the finite family J replaced by a countable one.

So if we let $K = \{x : Mf(x) > \lambda\}$, then $K \subset \{B_i\}_{i \in I}$, where the B_i range over all λ-dense balls. On each of these balls, we have that

$$\int_{B_j} |f| > t|B_j|$$

and such balls are of bounded radius as $f \in L_1$, so we can apply the Vitali covering lemma. So we have a countable subset $J \subset I$ for which

$$K \subseteq \bigcup_{j \in J} \subseteq \bigcup_{i \in I} 5B_i.$$

Now, we have that

$$V_{Mf}(\lambda) = |K| \leq sum_{j \in J}|5B_j| \leq 5^d \sum_{j \in J} |B_j| \leq 5^d \left(\int_{\bigcup_J B_j} |f| \right) \cdot \lambda^{-1} \lesssim \lambda^{-1} \|f\|_{L^1},$$

which is the bound that we wanted to prove. □

Remark: Stein in the 1970s removed the dependence on d in this theorem. So this theorem can be strengthened to say that $\|Mf\|_p \leq C(p)\|f|_p$.

1.2 Young's Inequality

Our next application of interpoliation is the following theorem:

Theorem 3 (Young's Inequality)**.**

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$$

if $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$.

Proof. Let us consider the operator $T_g f = f * g$ and use interpolation. Our boundary cases are $r = q$ and $r = \infty$. First, for $r = q$, we have that

$$\begin{aligned} \left\| \int f(y) g(x-y)\, dy \right\|_q &= \left\| \int f(y) g_y(x) \right\|_q \\ &\leq \int |f(y)| \|g_y\|_q \, dy \\ &= \|f\|_1 \|g\|_q, \end{aligned}$$

where the second line is a result of the integral form of Minkowski's inequality $\|f+g\|_q \leq \|f\|_q + \|g\|_q$. In the $r = \infty$ case, we have that

$$\left| \int f(y)g(x-y)\,dy \right| \leq \|f\|_p \|g(x-y)\|_q = \|f\|_p \|g\|_q,$$

by Holder's inequality if $\frac{1}{p} + \frac{1}{q} = 1$. Now, we can apply the general form of the Marcinkiewicz inequality, interpolating between the pairs $(q, 1)$ and $(\infty, \frac{q}{q-1})$ to get Young's inequality. ☐

1.3 Hardy-Littlewood-Sobolev

Our last application will be the Hardy-Littlewood-Sobolev inequality. Consider the following example:

$$T_\alpha f = f * (|x|^{-\alpha})$$

on $\mathbb{R}^d$. Notice that $|x|^{-\alpha}$ is "almost" in $L^{d/\alpha}$.

Theorem 4 (Hardy-Littlewood-Sobolev)**.** *If* $0 < \alpha < d, p > 1, \frac{1}{r} + 1 = \frac{1}{p} + \frac{\alpha}{q}$, *then*

$$\|T_\alpha f\|_{L^r} \lesssim \|f\|_{L^p}.$$

Proof Sketch. There are three main steps in this proof:

1. First, notice that
$$T_\alpha f(x) = c \int_0^\infty dr \cdot r^{d-\alpha-1} \fint_{B(x,r)} f.$$

2. Now we have two estimates
$$\left| \fint_{B(x,r)} f \right| \leq Mf(x) \tag{A}$$
and
$$\left| \fint_{B(x,r)} f \right| \sim r^{-d} \left| \int_{B(x,r)} 1 \cdot f \right| \leq r^{-d} \cdot r^{d/q} \|f\|_p. \tag{B}$$

3. We can combine the above estimates to get that
$$Tf_\alpha(x) \lesssim \int_0^\infty r^{d-\alpha-1} \min(A, B)\,dr.$$
Now, we want to decide when $A \leq B$ and then integrate. Doing this, we get that
$$Tf_\alpha(x) \lesssim Mf(x)^1 \cdot \|f\|_p^b$$

for $0 < a, b$ and $a + b = 1$ (by scaling). It follows then that

$$\int |Tf_\alpha|^r \lesssim \|f\|_p^{b\cdot r} \int Mf^{a\cdot r} \lesssim \|f\|_p^{b\cdot r} \|f\|_{a\cdot r}^{a\cdot r},$$

where the last inequality is a result of our result about the Hardy-Littlewood maximal function. To finish, we solve $p = a \cdot r$.

$\square$

2 Calderon-Zygmund

We're going to spend the next couple of classes talking about the Calderon-Zygmund theorem. First, let us consider the following theorem:

Theorem 5. *If $u \in C_c^\infty(\mathbb{R}^d)$, then for all $1 < p < \infty$,*

$$\|\partial_i \partial_j u\|_{L^p} \lesssim \|\Delta u\|_{L^p}.$$

Remark: We already proved this for $p = 2$, and we saw that it is false for $p = \infty$. For the $p = 2$ case, we wrote $u = \Delta u * \Gamma$ and then examined the second derivatives.

As an application, notice that if $u \in C_c^\infty(B_1)$ with $|\Delta u| \leq 1$, then

$$|\{x : |\partial_i \partial_j u(x)| > \lambda\}| \leq \|\partial_i \partial_j u\|_{L^p}^p \cdot \lambda^{-p}.$$

For $p = 2$, this implies that $|\cdot| \lesssim \lambda^{-2}$ and for general p, we have that $|\cdot| \lesssim C_p \lambda^{-p}$.

Related to this is the Calderon-Zygmund theorem, which we state here:

Theorem 6 (Calderon-Zygmund)**.** *If $Tf = f * K$, where*

$$|K(x)| \lesssim |x|^{-d}, \ |\partial K(x)| \lesssim |x|^{-d-1}, \ \int_{S_r} K = 0$$

for all r (initially, we'll suppose that $K \in C_c^0$), then $\|Tf\|_p \lesssim \|f\|_p, 1 < p < \infty$.

We'll start the proof of this theorem next class. It has four main steps:

1. An L^2 estimate $\|Tf\|_2 \lesssim \|f\|_2$
2. A weak L^1 estimate $V_{Tf}(\lambda) \cdot \lambda \lesssim \|f\|_{L^1}$
3. Now interpolation will give us an L^p estimate for $1 < p \leq 2$
4. A duality argument will then give us $2 \leq p < \infty$