18.156 Lecture Notes

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Today, we're starting the second unit in this course, which will be Fourier analysis. As an example of how Fourier analysis can be used to solve problems that a priori don't seem to be related to Fourier analysis, let us consider the **Gauss circle problem**. This problem asks us to estimate how many integer lattice points there are in a disk of radius R in $\mathbb{R}^2$. More formally, let

$$N(R) := \#\{(x,y) : x, y \in \mathbb{Z}, (x,y) \in B_R^2\}.$$

Then, a reasonable estimate for $N(R)$ is πR^2, the area of the circle of radius R. The error of this estimate is

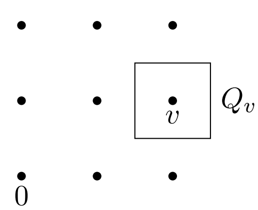
$$E(R) := N(R) - \pi R^2$$

and what we are interested in is a bound for $|E(R)|$.

First, let us show that we can find some bound for $|E(R)|$.

Proposition 1. *$|E(R)| \leq 100R$.*

Proof. For every $v \in \mathbb{Z}^2$, let Q_v be the unit squre in $\mathbb{R}^2$ centered at v.



Now,

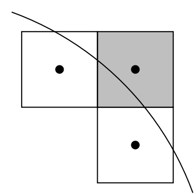
$$\begin{aligned}
N(R) &= \sum_{v \in \mathbb{Z}^2} \chi_{B_R}(v) \\
\pi R^2 &= \sum_{v \in \mathbb{Z}^2} \text{Area}(Q_v \cap B_R) \\
E(R) = N(R) - \pi R^2 &= \sum_{v \in \mathbb{Z}^2} (\chi_{B_R}(v) - \text{Area}(Q_v \cap B_R)).
\end{aligned}$$

But then,

$$|E(R)| \leq \#\{v : Q_v \cap \partial B_R \neq \emptyset\} \leq \#\{v : Q_v \subset B_{R+3} \backslash B_{R-3}\}.$$

□

But we could also have cancellation of overestimates and underestimates so it is reasonable to expect that we could get better than a linear bound. For example, in the following picture, the contribution to $E(R)$ from the shaded box is positive while the contribution from the unshaded boxes is negative. Perhaps we could exploit this cancellation.



To get some idea of what bounds on $|E(R)|$ we might expect to be possible, let us consider a random model. In this random model, $x_j \in [0,1]$ are uniformly distributed and independent, $j = 1, 2, \ldots, N$ ($N \sim R$). This represents the contribution to $E(R)$ of each lattice point where the contribution is nonzero (the points in a distance $\sqrt{2}/2$ neighborhood of the circle of radius R).

Proposition 2. $\mathbb{E}|\sum_{j=1}^N x_j| \leq CN^{1/2}$.

Proof.

$$\begin{aligned} LHS = \fint_{[-1,1]^N} \left|\sum_{j=1}^N x_j\right| dx &\leq \left(\fint \left(\sum_{j=1}^N x_j\right)^2 dx\right)^{1/2} \\ &= \left(\sum_{j_1,j_2} \fint x_{j_1} x_{j_2}\, dx\right)^{1/2} \\ &= \left(\sum_{j=1}^N \fint |x_j|^2\, dx\right)^{1/2} \lesssim N^{1/2}. \end{aligned}$$

Here, we're using Cauchy Schwarz in the first line and the orthogonality of the x_j to get the third line. □

The conjecture then is that for all $\epsilon > 0$, there exists C_ϵ such that $|E(R)| \leq C_\epsilon \cdot R^{\frac{1}{2}+\epsilon}$. What we will prove using tools from Fourier analysis is the following estimate, which is attributed to Sierpinski:

Theorem 3.

$$|E(R)| \lesssim R^{2/3}.$$

The best current bound of the form $|E(R)| \lesssim R^c$ is for $c = 131/208 \approx 0.63$, proven by Huxley in the early 2000s.

Let us now discuss the Fourier analysis setup in preparation for proving theorem 3. Let

$$f = \chi_{B_R^2}.$$

And for any $g \in L^1(\mathbb{R}^d)$, define the **periodization**

$$Pg(x) = \sum_{v \in \mathbb{Z}^d} g(x+v).$$

Then, $N(R) = Pf(0)$. If g is a $\mathbb{Z}^d$ periodic function on $\mathbb{R}^d$, then

$$\hat{g}(n) = \int_{[0,1]^d} g(x) e^{-2\pi i n \cdot x}\, dx.$$

We claim now that $\pi R^2 = \hat{Pf}(0)$. This is a result of the Poisson summation formula:

Theorem 4 (Poisson summation formula)**.** *If* $f \in L^1(\mathbb{R}^d), n \in \mathbb{Z}^d$*, then*

$$\hat{Pf}(n) = \hat{f}(n) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i n \cdot x}\, dx.$$

Proof. We have that

$$\begin{aligned}
\hat{Pf}(n) &= \int_{[0,1]^d} Pf(x) e^{-2\pi i n \cdot x}\, dx \\
&= \int_{[0,1]^d} \sum_{v \in \mathbb{Z}^d} f(x+v) e^{-2\pi i n \cdot x}\, dx \\
&= \sum_{v \in \mathbb{Z}^d} \int_{[0,1]^d} f(x+v) e^{-2\pi i n \cdot x}\, dx \\
&= \sum_{v \in \mathbb{Z}^d} \int_{[0,1]^d} f(x+v) e^{-2\pi i n \cdot (x+v)}\, dx,
\end{aligned}$$

since $n \in \mathbb{Z}^d$. So combining the sum and the integral, we have that

$$\hat{Pf}(n) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i n \cdot x}\, dx = \hat{f}(n).$$

□

Let us do some wishful thinking now. We could wish that

$$Pf(x) = \sum_{n\in\mathbb{Z}^2} \hat{Pf}(n) e^{2\pi i n\cdot x}.$$

(But this does not converge pointwise). Then,

$$N(R) = Pf(0) = \pi R^2 + \sum_{n\neq 0} \hat{Pf}(n)$$

and

$$|E(R)| \leq \sum_{n\in\mathbb{Z}^2\setminus\{0\}} |\hat{Pf}(n)|.$$

Here, we could with that this is $\leq C_\epsilon R^{\frac{1}{2}+\epsilon}$ (but unfortunately this sum happens to be infinite).

This leads us to the question of when does a Fourier series converge. We can begin to answer this through the following sequence of three theorems, with the first leading to the second leading to the third.

Theorem 5. *If $g \in L^2([0,1]^d)$, then $S_N g \to g$ in $L^2([0,1]^d)$. Here,*

$$S_N(g) = \sum_{|n|\leq N} \hat{g}(n) e^{2\pi i n x}.$$

Theorem 6. *If g is C^k on $\mathbb{R}^d$ and $\mathbb{Z}^d$ periodic, and $k > n$, then $S_N g \to g$ uniformly on C^0.*

Theorem 7. *If $\sum_n |\hat{g}(n)| < \infty, g \in C^0$, then $S_N g \to g$ uniformy in C^0.*

We also have the following question: how can we estimate $|\hat{g}(n)|$?

Proposition 8. *If g is $\mathbb{Z}^d$ periodic, $\|g\|_{C^k} \leq B$, then*

$$|\hat{g}(n)| \leq C(d,k)B \cdot |n|^{-k}.$$

Proof. We'll integrate by parts k times. For a fixed n, we'll integrate in x_j where j is chosen so that $|n_j| \leq \frac{1}{d}|n|$. Doing this, we see that

$$\begin{aligned}
\left|\int_{[0,1]^d} g(x) e^{-2\pi i n\cdot x}\, dx\right| &= \left|\int \partial_j g \cdot \frac{1}{-2\pi i n_j} e^{2\pi i n\cdot x}\, dx\right| \\
&= \left|\int \partial_j^k g \cdot \frac{1}{(-2\pi i n_j)^k} e^{2\pi i n\cdot x}\, dx\right| \\
&\leq |n_j|^k \int_{[0,1]^d} |\partial_j^k g| \\
&\lesssim |n|^{-k} \|\partial^k g\|_{C^0}.
\end{aligned}$$

$\square$

As a related question, we might ask if we could have a bound like $|\hat{g}(n)| \lesssim B|n|^{-\alpha}$ if $g \in C^\alpha$. Unfortunately, integration by parts doesn't work as well here, but we could use another method. Let us define $g_h(x) := g(x-h)$. Then, $|g(x) - g_h(x)| \lesssim h^\alpha$. So,

$$\begin{aligned}|\hat{g}(n) - \hat{g}_h(n)| &= \int (e^{-2\pi i n\cdot x} - e^{-2\pi i n\cdot (x+h)}) g(x)\, dx \\ &= (1 - e^{-2\pi i n \cdot h})\hat{g}(n).\end{aligned}$$

But we also have the bound that

$$|\hat{g}(n) - \hat{g}_h(n)| \leq \int_{[0,1]^d} |g(x) - g(x+h)|\, dx \lesssim h^\alpha.$$

Combining these, we have that

$$|\hat{g}(n)| \leq |1 - e^{-2\pi i n\cdot h}|^{-1} h^\alpha,$$

and we can optimize our choice of h to get the bounds that we want.

Perhaps we're not satisfied by the integration by parts proof of the previous proposition and want a way of visualizing why smoothness of the function g would lead to decay of the Fourier coefficients $\hat{g}(n)$. Let us consider a smooth, slowly varying function g in one dimension and a large n. Then, just looking at the real part for visualization purposes, $\mathrm{Re}(g(x)e^{-2\pi i n x})$ looks like a scaled cosine function with some error. The "positive" and "negative" bumps then almost cancel and we would expect more cancellation for larger n.

More formually, let us subdivide $[0,1]$ into intervals I_j of length $1/n$. Then,

$$\begin{aligned}\left|\int_0^1 g(x)e^{-2\pi i n\cdot x}\, dx\right| &= \left|\sum_j \int_{I_j} (g(x) - g(x_j))e^{-2\pi i n \cdot x}\, dx\right| \\ &= \sum_j \int_{I_j} |g(x) - g(x_j)|\, dx,\end{aligned}$$

and if n is larger, then we can bound $|g(x) - g(x_j)|$ better.

Our next goal will be to estimate $|\hat{Pf}(n)|$. Let us do the first step now. For $f = \chi_{B_R}$,

$$|\hat{Pf}(n)| = \left|\int_{B_R} e^{-2\pi i n\cdot x}\, dx\right|$$

and by rotational invariance, we then have that

$$|\hat{Pf}(n)| = \left|\int_{B_R} e^{-2\pi i |n| x_1}\, dx_1\, dx_2\right| = \left|\int_{-R}^{R} 2\sqrt{R^2 - x_1^2}e^{-2\pi i|n|x_1}\, dx_1\right|.$$