

Course Description for Real Analysis, Math 156

In this class, we will study elliptic PDE, Fourier analysis, and dispersive PDE.

Here is a quick summary of the topics we will study. They're described in more detail below. We will study variable-coefficient elliptic PDE, looking at regularity estimates like the Schauder inequality and the De Giorgi-Nash-Moser inequality. We will use these estimates to solve elliptic PDE, starting with linear PDE and then turning to non-linear PDE. We will study applications of Fourier analysis, including a little combinatorics/number theory as well as applications to PDE. We will use Fourier and harmonic analysis to prove L^p estimates for operators such as the Calderon-Zygmund inequality for the Laplacian. In dispersive PDE, we will study the Schrodinger equation. We again look at estimates, and build up to the Strichartz inequality. We will use the Strichartz inequality to study solutions of non-linear dispersive PDE.

We will also practice different strategies for proving estimates. For instance, when is it a good idea to break a sum into pieces, and which pieces should we use? When is it good to use Holder's inequality? When do we need to exploit cancellation, and how do we prove that cancellation happens? When we look at a complex expression, how do we decide which terms are important and which are less important?

If you haven't taken Math 155, here are the prerequisites you should have to take the course. You should know about L^p spaces, and a little bit about Sobolev spaces. You should be familiar with the Fourier transform, with the Fourier inversion theorem and the Plancherel theorem.

Here is a more detailed description of the topics of the class.

Elliptic PDE

We will study variable-coefficient elliptic PDE and non-linear elliptic PDE. These PDE generalize the Laplace equation $\Delta u = 0$ defining harmonic functions. Many important properties of harmonic functions hold in much wider generality, and this is our main topic. Let's recall two important properties of harmonic functions. Let $B_r \subset \mathbb{R}^n$ denote the ball of radius r around the origin.

Theorem 1. (Regularity) *If $u : B_1 \rightarrow \mathbb{R}$ is a harmonic function, then for any $k \geq 1$,*

$$\|u\|_{C^k(B_{1/2})} \leq C_{k,n} \|u\|_{C^0(B_1)}.$$

Theorem 2. (Solution to the Dirichlet problem) *If $g : \partial B_1 \rightarrow \mathbb{R}$ is a smooth function, then there is a unique function $u : B_1 \rightarrow \mathbb{R}$ satisfying*

- $\Delta u = 0$.
- u restricted to ∂B_1 is equal to g .

A variable coefficient elliptic operator is an operator L of the form

$$Lu(x) := \sum_{i,j=1}^n a_{ij}(x) \partial_i \partial_j u(x), \quad (1)$$

where $a_{ij}(x)$ is a positive definite symmetric matrix depending on x . If a_{ij} is constant in x , then after a change of variables, the operator L becomes the Laplacian Δ . But for variable $a_{ij}(x)$, we get a much wider class of operators. These operators play a big role in the PDE of differential geometry.

There are two main approaches to study variable coefficient operators. The perturbative approach studies Lu as a small variation of the Laplacian. This works if we know some regularity of the coefficients a_{ij} , for example if a_{ij} is Holder C^α for some $\alpha > 0$. In that case, on small balls, a_{ij} is almost constant, and we can think of Lu as a small modification of the Laplacian. With this philosophy, Schauder was able to prove the following regularity estimate.

Theorem 3. (*Schauder regularity estimate*) *Suppose that L is as above, with $a_{ij} \in C^\alpha$, and the eigenvalues of a_{ij} are pinched in the range $0 < \lambda \leq a_{ij} \leq \Lambda$. Then there is a constant C so that, for any solution $Lu = 0$ defined on B_1 ,*

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C \|u\|_{C^0(B_1)}.$$

If we have no estimate on the regularity of a_{ij} , then L does not resemble the Laplacian in any real sense, and we cannot study it as a small perturbation of Δ . But even in this dramatically more general setting, some regularity estimate still holds. Such regularity was first established by De Giorgi-Nash-Moser. Their theorem applies to operators written in a slightly different way, called divergence form:

$$Lu := \sum_{i,j=1}^n \partial_i (a_{ij} \partial_j u). \quad (2).$$

Theorem 4. (*De Giorgi-Nash-Moser*) *Suppose that L is as defined by equation (2). Suppose that the eigenvalues of a_{ij} are pinched in the range $0 < \lambda \leq a_{ij} \leq \Lambda$. Then there are constants $C, \alpha > 0$ depending only on λ, Λ, n so that the following holds. For any solution $Lu = 0$ defined on B_1 ,*

$$\|u\|_{C^\alpha(B_{1/2})} \leq C \|u\|_{C^0(B_1)}.$$

Next we will discuss solving elliptic PDE, proving generalizations of Theorem 2 above. In Theorem 2, the solution u is given by a formula. But for other operators, there is no way to find a formula for the solution. A major topic of analysis is how to prove that there are solutions to an equation without being able to describe them

by a closed formula. All different approaches to the problem involve estimates like the ones we've been talking about. We will try to explain why estimates help to find solutions of a PDE. Basically, since there is no closed formula for the solution, we would like to find a procedure that approximates the solution more and more accurately, and prove that the approximations converge to the solution. To prove that the approximations converge and/or to prove that the approximations nearly solve the equation are both problems about estimates, and these estimates turn out to be closely connected to the estimates above.

In particular we will focus on the method of continuity. In this method, we consider a 1-parameter family of problems, depending on a parameter $t \in [0, 1]$. We arrange that when $t = 0$, the problem is one that we already understand, like the Laplace equation on the ball, and at $t = 1$, the problem is the problem of interest. As t changes from 0 to 1, we need to prove that there is a unique solution at each t . One of the main worries is that the solution at time t may become wilder and wilder as $t \rightarrow t_0 < 1$, and there may be no solution for $t \geq t_0$. This scenario can be ruled out if we can prove appropriate regularity estimates. Using this strategy we will solve the Dirichlet problem for $Lu = 0$ and also for some non-linear PDE.

The minimal surface equation is an interesting non-linear elliptic PDE. Solutions of the minimal surface equation model soap films. They play an important role in differential geometry. We will study the minimal surface equation, proving regularity estimates and discussing the Dirichlet problem.

Applications of Fourier analysis

The Fourier transform is a remarkably versatile tool, with applications in number theory, combinatorics, geometry, and PDE. We will study some applications with an eye towards learning to use the Fourier transform as a tool.

The Fourier transform allows us to take an almost arbitrary function $f : \mathbb{R}^n \rightarrow \mathbb{C}$ (with some mild conditions on the growth of f) and write it as a linear combination of waves $e^{i\omega x}$ at different frequencies ω . Given a problem about a function f , we can study the problem using this decomposition. When is it useful to do this? Why is it useful to do this?

The functions $e^{i\omega x}$ are eigenfunctions of all the partial derivative operators ∂_j . More generally, they are eigenfunctions for translation operators. For any $v \in \mathbb{R}^n$, define the operator T_v by $T_v f(x) := f(x + v)$. Then we see that $T_v(e^{i\omega x}) = e^{i\omega v}(e^{i\omega x})$. Decomposing f into these eigenfunctions can be a good way to study f in a problem involving derivatives or translations or translational symmetry.

Here is a (perhaps surprising) example. The Gauss circle problem studies the number of integer points in the disk of radius R in \mathbb{R}^2 . Let $N(R)$ denote this number. A reasonable first guess is that $N(R)$ is close to the area of the circle

πR^2 . Let $E(R) = N(R) - \pi R^2$ be the error in this guess. The Gauss circle problem asks, how large is $E(R)$? Informed conjecture and computer simulation suggest that $E(R) \leq C_\epsilon R^{(1/2)+\epsilon}$ for any $\epsilon > 0$ - this is a deep open problem of number theory. There is an elementary geometry argument showing that $E(R) \leq 10R$. This estimate was improved by Sierpinski to $E(R) \leq CR^{2/3}$ using Fourier analysis. Why does it have anything to do with Fourier analysis? The problem involves the lattice of integers \mathbb{Z}^2 . If we perturb the set of lattice points by shifting them each a small amount in an arbitrary direction, then the error term could be as large as $\sim R$. The Gauss circle problem depends delicately on the integers being exactly where they are. How can we use this structure? What properties of the integer lattice can we bring to bear on this estimate? Sierpinski's argument is based on the fact that the integer lattice is invariant under many translations, and this translational symmetry brings Fourier analysis into the game.

Fourier analysis specializes in proving estimates based on symmetry. Often these estimates involve cancellation. Suppose we want to estimate a sum of many terms, with positive and negative contributions. This can be a delicate problem. For an instance, the Gauss circle problem can be easily rewritten as an estimate for a sum $\sum_i a_i$ of R terms, where each a_i has norm at most 1. The triangle inequality immediately gives $|\sum a_i| \leq R$, but we are hoping to prove a better estimate like $|\sum a_i| \lesssim R^{2/3}$ or even $R^{(1/2)+\epsilon}$. If we estimate the total positive contribution up to a factor of 1.001, and we estimate the total negative contribution up to a factor 1.001, it will still be hard to tell how close they are to canceling. Trying to estimate each contribution even more accurately is very difficult. When we are able to prove cancellation, it's usually because there is some symmetry that causes the positive terms to (almost) balance the negative terms. Fourier analysis is an important tool for carrying out this type of estimate by symmetry.

Besides estimates in combinatorics/number theory like the Gauss circle problem, we will mainly focus on L^p estimates for operators, especially operators appearing in PDE. For example, we will prove the Calderon-Zygmund estimate for the Laplacian, which also plays an important role in elliptic regularity.

Theorem 5. (*Calderon-Zygmund*) *If $1 < p < \infty$, and if u is a smooth compactly supported function $u : \mathbb{R}^n \rightarrow \mathbb{R}$, then*

$$\|\partial_i \partial_j u\|_{L^p} \leq C(p, n) \|\Delta u\|_{L^p}.$$

A crucial tool in estimating L^p norms of operators is interpolation. Here is a special case:

Theorem 6. (*Special case of interpolation*) *Suppose that $1 \leq p < q \leq \infty$. Suppose that T is a linear operator that obeys*

- $\|Tf\|_p \leq C_1 \|f\|_p$ and
- $\|Tf\|_q \leq C_2 \|f\|_q$.

Then for any r in the range $p < r < q$, there is a constant C_r so that

$$\|Tf\|_r \leq C_r \|f\|_r.$$

We will prove the Marcinkiewicz interpolation theorem, which is more general than this special case in several ways. In general, interpolation allows us to deduce new L^p estimates from old L^p estimates. It is a fundamental tool in understanding the L^p estimates obeyed by an operator.

We'll also discuss why we want to prove L^p estimates about operators. Given a linear operator T , it's interesting to find all the pairs (p, q) so that $\|Tf\|_p \leq C \|f\|_q$ for some finite C . What kind of information about T does this list of pairs (p, q) encode? We'll talk about this in the course, and we'll also address it a little bit more in the next section on dispersive PDE.

Dispersive PDE

The wave equation and the Schrodinger equation are key examples of dispersive equations. They are important in physics: the wave equation models sound waves, and to some extent light waves. Other kinds of waves, like water waves, are more complex, but understanding the wave equation is still an important tool for approaching these problems. The Schrodinger equation models a quantum mechanical particle.

The equations apply to a function u that depends on a space variable x in \mathbb{R}^n and a time variable $t \in \mathbb{R}$. (The case $n = 3$ is particularly interesting for physics, but all dimensions n can be interesting mathematically.) They enjoy important conservation laws like the conservation of energy for the wave equation. The Schrodinger equation has a particularly simple conservation law: if $u(x, t)$ solves the Schrodinger equation, then the integral $\int_{\mathbb{R}^3} |u(t, x)|^2 dx$ is independent of t . The conservation of energy for the wave equation is similar with a more complex formula involving derivatives of u .

The function $|u(t, x)|^2$ is called the mass density for the Schrodinger equation. The total mass is conserved, but the way that the mass density is arranged changes over time. The Schrodinger equation is called dispersive because the mass density tends to disperse over time. At a single time, say $t = 0$, the mass may be concentrated into an arbitrarily small region, but it cannot stay concentrated over a substantial time interval. Over a large time interval, the mass is forced to spread out. This phenomenon can be described precisely using L^p norms. The L^2 norm $\|u(\bullet, t)\|_{L_x^2}$ is independent of time, and let's suppose as an example that this norm is 1. Now for $p > 2$, if the mass is very concentrated at time t , then $\|u(\bullet, t)\|_{L_x^p}$ will be much larger than 1. Vice versa, if $\|u(\bullet, t)\|_{L_x^p}$ is small, then the mass must be spread out. So the

dispersion can be described by a space-time estimate on $\|u\|_{L^p(\mathbb{R}^n \times \mathbb{R})}$. If we have a bound on $\|u\|_{L^p(\mathbb{R}^n \times \mathbb{R})}$, then at most times t , we have a bound on $\|u(\bullet, t)\|_{L_x^p}$, and we see that at most times the mass is dispersed.

The Strichartz estimate is a space-time L^p estimate for solutions of the Schrodinger equation.

Theorem 7. (*Strichartz*) *Suppose that $u(x, t)$ is a solution of the Schrodinger equation on $\mathbb{R}^d \times \mathbb{R}$. Then for $p = \frac{2(n+2)}{n}$,*

$$\|u\|_{L^p(\mathbb{R}^n \times \mathbb{R})} \leq C_n \|u(\bullet, 0)\|_{L_x^2}.$$

This important result gives a lot of information about the way that solutions to the Schrodinger equation disperse. The exponent $p = \frac{2(n+2)}{n}$ is the unique exponent that makes this inequality hold. We will prove the Strichartz estimate using the tools from Fourier and harmonic analysis that we studied earlier.

Finally, we will study some simple non-linear dispersive PDE's. The existence of solutions for the non-linear PDE depends on proving estimates. Estimates for the linear PDE, such as the Strichartz inequality, are a crucial building block to prove estimates about non-linear PDE.