Decoupling, Problem set 4

This is the last problem set for the class. We will focus on digesting the proof of the sharp decoupling theorem (for the paraboloid).

When studying a complicated proof, it can be helpful to think about how it plays out in some (fairly) simple examples. In the first problem, we will consider an important class of examples for the decoupling problem.

1. We consider the decoupling problem for the paraboloid in n dimensions. Here is a class of examples that we will analyze. To describe these examples, we will have to discuss caps of various scales. We write θ_1 for an $R^{-1/2}$-cap, θ_2 for an $R^{-1/4}$-cap, etc.

Suppose that for each $R^{1/2}$-cap θ_1 in the paraboloid, f_{θ_1} consists of W_1 wave packets in the ball B_R, each with amplitude H_1. Suppose also that there are $V_1 \sim W_1^{\frac{n}{n-1}}$ $R^{1/2}$-boxes Q_1 in B_R which are each contained in one wave packet for every cap θ_1. (Draw a picture of how this may occur. If you're latexing your solutions, you don't have to turn the picture in if it's not convenient.) Suppose that $\|f\|_p$ is dominated by the contribution from these V_1 $R^{1/2}$-boxes.

Next, we look inside each box Q_1 from the first paragraph. Suppose that inside each box Q_1, for each $R^{1/4}$-cap θ_2, f_{θ_2} consists of W_2 wave packets (of length $R^{1/2}$ and radius $R^{1/4}$), each with amplitude H_2. Suppose also that there are $V_2 \sim W_2^{\frac{n}{n-1}}$ $R^{1/4}$-boxes $Q_2 \subset Q_1$ which are each contained in one wave packet for every cap θ_2. Suppose that $\|f\|_{L^p(Q_1)}$ is dominated by the contribution from these V_2 $R^{1/4}$ boxes $Q_2 \subset Q_1$, and so $\|f\|_{L^p}$ is dominated by the contibutions of these $Q_1 Q_2$ $R^{1/4}$-boxes. (Draw a second picture showing the wave packets and boxes at the two scales. Again, you don't have to turn it in if it's not convenient.)

We continue this way through s scales, where $s \sim \log\log r$, so that the final caps θ_s have size ~ 1 and the final boxes Q_s have size around 1.

a.) Using local orthogonality, prove that

$$H_j \lesssim R^{\frac{n-1}{2^j}} W_j^{-1/2} H_{j-1}.$$

b.) Next consider $p = \frac{2(n+1)}{n-1}$, the sharp exponent in decoupling, and estimate the decoupling ratio

$$\frac{\|f\|_{L^p}}{(\sum_{\theta_1} \|f_{\theta_1}\|_{L^p}^2)^{1/2}}.$$

Prove the bound

$$\frac{\|f\|_{L^p}}{(\sum_{\theta_1} \|f_{\theta_1}\|_{L^p}^2)^{1/2}} \lesssim (\log R)^{O(1)} \left(\frac{W_1}{W_2 ... W_s} \right)^{\frac{1}{2(n+1)}}.$$

In particular, this shows that the decoupling ratio is $\lesssim R^\epsilon$ whenever $W_1 \le W_2 ... W_s$.

c.) Next we consider a different way of estimating this ratio. This time, we bring into play decoupling of smaller caps, namely:

$$\|f_{\theta_2}\|_{L^p} \lesssim D_p(R^{1/2}) \left(\sum_{\theta_1 \subset \theta_2} \|f_{\theta_1}\|_{L^p}^2 \right)^{1/2}.$$

Use this fact to estimate the ratio

$$\frac{(\sum_{\theta_2} \|f_{\theta_2}\|_{L^p}^2)^{1/2}}{(\sum_{\theta_1} \|f_{\theta_1}\|_{L^p}^2)^{1/2}}.$$

Then use the method above to estimate the ratio

$$\frac{\|f\|_{L^p}}{(\sum_{\theta_2} \|f_{\theta_2}\|_{L^p}^2)^{1/2}}.$$

Combining these estimates, prove the bound

$$\frac{\|f\|_{L^p}}{(\sum_{\theta_1} \|f_{\theta_1}\|_{L^p}^2)^{1/2}} \lesssim (\log R)^{O(1)} D_p(R^{1/2}) \left(\frac{W_2}{W_3 ... W_s}\right)^{\frac{1}{2(n+1)}}.$$

Give an example (of $W_1, W_2, ... W_s$) when the bound from 1c is better than the bound from 1b, and an example when the bound from 1b is better than the bound from 1c.

d.) By the same idea as in the proof of 1c., for any σ in the range $1 \le \sigma \le s$, we can show that

$$\frac{\|f\|_{L^p}}{(\sum_{\theta_1} \|f_{\theta_1}\|_{L^p}^2)^{1/2}} \lesssim (\log R)^{O(1)} D_p(R^{1-2^{1-\sigma}}) \left(\frac{W_\sigma}{W_{\sigma+1} ... W_s}\right)^{\frac{1}{2(n+1)}}.$$

Show that for any $\epsilon > 0$ and any fixed $W_1, ..., W_s$, we can choose $\sigma \le C\epsilon^{-1}$ so that

$$\frac{\|f\|_{L^p}}{(\sum_{\theta_1} \|f_{\theta_1}\|_{L^p}^2)^{1/2}} \lesssim D_p(R^{1-2^{1-\sigma}}) R^{\epsilon 2^{1-\sigma}}.$$

If $D_p(R)$ was realized by a function f of the type considered in this question, we would get a recursive inequality: for any $\epsilon > 0$

$$D_p(R) \lesssim \max_{1 \le \sigma \le C\epsilon^{-1}} D_p(R^{1-2^{1-\sigma}}) R^{2\epsilon 2^{1-\sigma}}. \tag{$*$}$$

Note that $(*)$ implies that $D_p(R) \lesssim R^{3\epsilon}$.

This begs the question, is the ratio $D_p(R)$ approximately realized by a function f of the type considered in this question? What other kinds of functions could arise? What else could go wrong? We will explore this some in Problem 2 and talk about it more in class.

2. Now we consider a generalization of the scenario from Problem 1.

a.) Suppose again that for each $R^{1/2}$-cap θ_1 in the paraboloid, f_{θ_1} consists of W_1 wave packets in the ball B_R, each with amplitude H_1. Suppose now that there are $V_1 \gtrsim W_1^{\frac{n}{n-1}}$ $R^{1/2}$-boxes Q_1 in B_R which make a dominant contribution to f. Suppose that $V_1 = (\beta_1 W_1)^{\frac{n}{n-1}}$, and suppose that each box Q_1 is contained in a wave packet of f_{θ_1} for a fraction $1/\beta_1$ of caps θ_1.

And similarly, at smaller spatial scales, suppose that inside each box Q_{j-1}, for each $R^{2^{-j}}$-cap θ_j, f_{θ_j} consists of W_j wave packets (of length $R^{2^{1-j}}$ and radius $R^{2^{-j}}$), each with amplitude H_j. And suppose that there are $V_j \sim (\beta_j W_j)^{\frac{n}{n-1}}$ boxes $Q_j \subset Q_{j-1}$ which each lie in a wave packet of f_{θ_j} for a fraction β_j^{-1} of caps θ_j.

Estimate the ratio $\frac{\|f\|_{L^p}}{(\sum_{\theta_1} \|f_{\theta_1}\|_{L^p}^2)^{1/2}}$ and show that it is smaller than the upper bounds given in Problem 1.

b.) Is it possible to arrange far more than $W_1^{\frac{n}{n-1}}$ cubes Q_1 so that each cube Q_1 lies in a wave packet from each f_{θ_1}? Explain your answer.

c.) Suppose β_1 is large. Is it possible to arrange far more than $(\beta_1 W_1)^{\frac{n}{n-1}}$ cubes Q_1 so that each cube Q_1 lies in a wave packet of f_{θ_1} for at least a fraction $(1/\beta_1)$ of the caps θ_1? You may not be able to answer this question, but does it remind you of any other questions we have seen? If such an arrangement is possible, can we say anything interesting/useful about it?

d.) In what other ways could a function f be different from the examples we have considered so far? Pick out one or two issues that seem the most significant to you.

3. Last problem set, you wrote an outline of the proof of multilinear restriction. Read over your outline, and then try to prove multilinear restriction following your outline. How did it go? Did any tricky points come up when you tried to carry this out?

4. Write an outline of the proof of the decoupling theorem for the paraboloid for $2 \leq p \leq \frac{2n}{n-1}$ (the non-sharp decoupling theorem). The outline should be a few steps, mostly in words or short equations. On the one hand, the outline should be a lot shorter than the whole proof. On the other hand, you should try to include "the main ideas". You might imagine that in a few weeks, you would try to reconstruct the proof just based on this outline. What is the key information that you should record for yourself?