18.118 Topics in Analysis Problem Set 3

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Problem 1. Write an outline of the proof of multilinear restriction.

Proof. The statement is

$$\|\prod_{j=1}^{n}|f_j|^{\frac{1}{n}}\|_{L^p_{\text{avg}}(B_R)} \lesssim R^\epsilon \prod_{j=1}^{n}\|f_j\|^{\frac{1}{n}}_{L^2_{\text{avg}}(B_R)} \qquad p=\frac{2n}{n-1}$$

where $f=\sum_{j=1}^n f_j$, supp $\widehat{f_j}\subset N_{1/R}\Sigma_j$ and Σ_j are hypersurfaces with normal vector almost parallel to x_j-axis.
Take p-th power on both sides and induction on scales. Tile B_R using balls of radius $R^{1/2}$. The averaged integral over B_R on the left hand side is approximately the average over all balls of radius $R^{1/2}$ in the tiling of the averaged integral over one ball of radius $R^{1/2}$. Suppose the statement is true at scale $R^{1/2}$, use local orthogonality and local constancy at scale $R^{1/2}$, and put everything back to integration over B_R. Then use Multilinear Kakeya and the reverse direction of local orthogonality. □

Problem 2. Let E_S denote the extension operator for the surface $S\subset\mathbb{R}^n$,

$$E_S\phi(x)=\int_S e^{2\pi i\omega\cdot x}\phi(\omega)\,d\text{vol}_S(\omega).$$

Let $\mathrm{E}_{S,p}(R)$ be the best constant in the inequality $\|E_S\phi\|_{L^p(B_R)}\le C\|\phi\|_{L^p(S)}$. Let $\mathrm{E}_{n,p}(R)$ be the maximum of $\mathrm{E}_{S,p}(R)$ over all S with diameter 1, C^3-norm at most 10 and principal curvatures between 1/10 and 10 ("well-curved"). Suppose that S is well-curved in this sense and that $\tau\subset S$ is a K^{-1} cap. Show that

$$\|E_s\phi_\tau\|_{L^p(B_R)}\le C(n)\mathrm{E}_{n,p}(\frac{R}{K})K^{\frac{2n}{p}-(n-1)}\|\phi_\tau\|_{L^p(S)}.$$

Proof. Let L be the linear change of variables that looks like stretching $\omega_1,...,\omega_{n-1}$ by K and stretching ω_n by K^2 (if S is a paraboloid, then L maps τ to a paraboloid). Then L maps τ to a well-curved surface S', and

$$c(n)K^{n+1}\le|\det L|\le C(n)K^{n+1}, \tag{1}$$

while restricting L on τ, the determinant $\sim K^{n-1}$. We can do this because the bounds of principal curvatures and C^3 norm allow us to do second order approximation of the surface locally. We have

$$\begin{aligned}
E_S\phi_\tau(x) &= \int_\tau e^{2\pi i\omega\cdot x}\phi(\omega)\,\mathrm{dvol}_S\omega \\
&\sim \int_{L(\tau)} e^{2\pi i L^{-1}\omega'\cdot x}\phi(L^{-1}\omega')K^{-(n-1)}\,\mathrm{dvol}_{L(\tau)}\omega' \\
&= K^{-(n-1)}\int_{S'} e^{2\pi i\omega'\cdot(L^{-1}x)}\psi(\omega')\,\mathrm{dvol}_{S'}\omega' \\
&= K^{-(n-1)}E_{S'}\psi(L^{-1}x)
\end{aligned}$$

where $\psi(\omega') := \phi(L^{-1}\omega')$ and the $\sim$ is because of the inverse version of (1). Hence, we have

$$\begin{aligned}
\|E_S\phi_\tau\|^p_{L^p(B_R)} &= K^{-(n-1)p}\int_{B_R}|E_{S'}\psi(L^{-1}x)|^p\,dx \\
&= K^{-(n-1)p}\int_{L^{-1}(B_R)}|E_{S'}\psi(x')|^p|\mathrm{det}L|\,dx' \\
&\sim K^{-(n-1)p}\int_{B_{R/K}}|E_{S'}\psi(x')|^pK^{n+1}\,dx' \\
\implies \qquad \|E_S\phi_\tau\|_{L^p(B_R)} &\sim K^{\frac{n+1}{p}-(n-1)}\|E_{S'}\psi\|_{L^p(B_{R/K})} \\
&\leq K^{\frac{n+1}{p}-(n-1)}\mathrm{E}_{n,p}(\frac{R}{K})\|\psi\|_{L^p(S')}.
\end{aligned}$$

Also note that

$$\begin{aligned}
\|\psi\|^p_{L^p(S')} &= \int_{S'}|\psi(\omega)|^p\,\mathrm{dvol}_{S'}\omega \\
&= \int_{L(\tau)}|\phi(L^{-1}\omega)|^p\,\mathrm{dvol}_{L(\tau)}\omega \\
&\sim \int_\tau |\phi(\omega')|^pK^{n-1}\,\mathrm{dvol}_\tau\omega' \\
\implies \qquad \|\psi\|_{L^p(S')} &\sim K^{\frac{n-1}{p}}\|\phi_\tau\|_{L^p(S)}.
\end{aligned}$$

Putting things together, we have

$$\begin{aligned}
\|E_S\phi_\tau\|_{L^p(B_R)} &\leq C(n)K^{\frac{n+1}{p}-(n-1)}\mathrm{E}_{n,p}(\frac{R}{K})K^{\frac{n-1}{p}}\|\phi_\tau\|_{L^p(S)} \\
&\leq C(n)K^{\frac{2n}{p}-(n-1)}\mathrm{E}_{n,p}(\frac{R}{K})\|\phi_\tau\|_{L^p(S)}
\end{aligned}$$

□

Problem 3. Let $\mathrm{ME}_{n,p}(R)$ denote the best constant so that

$$\Big\| \prod_{j=1}^{n} |E_{S_j}\phi_j|^{\frac{1}{n}} \Big\|_{L^p(B_R)} \le C \prod_{j=1}^{n} \|\phi_j\|_{L^p(S_j)}^{\frac{1}{n}}$$

whenever S_j are surfaces with diameter at most 1, C^3-norm at most 10, and normal vector almost parallel to the x_j-direction. Show that when $n = 3$,

$$\mathrm{E}_{3,p}(R) \lesssim K^{O(1)}\mathrm{ME}_{3,p}(R) + D_{2,p}(K^2)\mathrm{E}_{3,p}(\frac{R}{K})K^{\frac{5}{p}-\frac{3}{2}}. \tag{2}$$

We know from class that $D_{2,p}(K^2) \lesssim K^{\epsilon}$ for $2 \le p \le 4$. The multilinear restriction theorem leads directly to good bounds for $\mathrm{ME}_{n,p}(R)$. Using these facts and equation (2), prove that $\mathrm{E}_{3,p}(R) \lesssim R^{\epsilon}$ for $p \ge 10/3$.

Proof. We partition S into K^{-1}-caps τ so there are $\sim K$ such τ's, and tile B_R with balls of radius K^2. Define significant part $S(B_{K^2})$, "broad" and "narrow" balls of radius K^2 as in the lecture. We need to show that

$$\|E_S\phi\|_{L^p(B_R)} \lesssim \Big(K^{O(1)}\mathrm{ME}_{3,p}(R) + D_{2,p}(K^2)\mathrm{E}_{3,p}(\frac{R}{K})K^{\frac{5}{p}-\frac{3}{2}}\Big)\|\phi\|_{L^p(S)}$$

for all well-curved $S \subset \mathbb{R}^3$. We separate the broad part and narrow part,

$$\begin{aligned}
&\|E_S\phi\|_{L^p(B_R)}\\
\lesssim&\Big(\sum_{\substack{B_{K^2}\subset B_R\\ \text{broad}}} \int_{B_{K^2}} |E_S\phi(x)|^p\,dx\Big)^{\frac{1}{p}} + \Big(\sum_{\substack{B_{K^2}\subset B_R\\ \text{narrow}}} \int_{B_{K^2}} |E_S\phi(x)|^p\,dx\Big)^{\frac{1}{p}}\\
=&\|\text{broad part}\|_{L^p(B_R)} + \|\text{narrow part}\|_{L^p(B_R)}
\end{aligned}$$

by triangle inequality ($\lesssim$ is because this is not a perfect tiling but the error is small).

Narrow estimate: if B_{K^2} is narrow, then there exists a hyperplane Π^* such that $\forall\,\tau \in S(B_{K^2})$, $\mathrm{Angle}(\mathrm{nor}(\tau), \Pi^*) \lesssim K^{-1}$.

$$\sum_{\substack{B_{K^2}\subset B_R\\ \text{narrow}}} \int_{B_{K^2}} |E_S\phi(x)|^p\,dx \sim \sum_{\substack{B_{K^2}\subset B_R\\ \text{narrow}}} \int_{B_{K^2}} \Big|\sum_{\tau\in S(B_{K^2})} E_S\phi_\tau(x)\Big|^p\,dx$$

For a single narrow B_{K^2}, rotate the coordinate axes if necessary so that $\mathrm{nor}(\tau)$ almost parallel to the 3rd direction for each $\tau \in S(B_{K^2})$. By Fubini's theorem,

$$\int_{B_{K^2}} \Big|\sum_{\tau\in S(B_{K^2})} E_S\phi_\tau(x)\Big|^p\,dx = \int_{-K^2}^{K^2} \Big\|\sum_{\tau\in S(B_{K^2})} E_S\phi_\tau(x',t)\Big\|^p_{L^p(B_{K^2}\cap\Pi_t)}\,dt$$

By giving S a parametrization, for fixed t, as a function of $x' \in \mathbb{R}^2$, $E_S\phi_\tau(x',t)$ is the Fourier inverse of some function that have compact support in $\mathbb{R}^2$, and since S is well-behaved we can assume that support is also well-behaved so that Decoupling theorem can be applied. By the lemma in the lecture preceding the narrow estimate,

$$\begin{aligned}
&\int_{B_{K^2}} \Big|\sum_{\tau\in S(B_{K^2})} E_S\phi_\tau(x)\Big|^p\,dx \\
\lesssim &\int_{\mathbb{R}} D_{2,p}(K^2)^p\Big(\sum_{\tau\in S(B_{K^2})} \|E_S\phi_\tau\|^2_{L^p(B_K\cap\Pi_t)}\Big)^{\frac{p}{2}}\,dt \qquad \text{drop the weight}\\
\leq &D_{2,p}(K^2)^p\int_{\mathbb{R}}\Big(\Big[\sum_{\tau\in S(B_{K^2})}\|E_S\phi_\tau\|^{2\cdot\frac{p}{2}}_{L^p(B_K\cap\Pi_t)}\Big]^{\frac{2}{p}}\Big[\sum_{\tau\in S(B_{K^2})}1\Big]^{1-\frac{2}{p}}\Big)^{\frac{p}{2}}\,dt\\
\lesssim &D_{2,p}(K^2)^p\int_{\mathbb{R}}\sum_\tau \|E_S\phi_\tau\|^p_{L^p(B_{K^2}\cap\Pi_t)}K^{\frac{p}{2}-1}\,dt.
\end{aligned}$$

The $\leq$ in the thid line is due to Hölder's inequality. Summing over all $B_{K^2}\subset B_R$, we have

$$\begin{aligned}
\sum_{\substack{B_{K^2}\subset B_R\\ \text{narrow}}}\int_{B_{K^2}}|E_S\phi(x)|^p\,dx &\lesssim D_{2,p}(K^2)^pK^{\frac{p}{2}-1}\sum_\tau\int_{\mathbb{R}}\|E_S\phi_\tau\|^p_{L^p(B_R\cap\Pi_t)}\,dt\\
&= D_{2,p}(K^2)^pK^{\frac{p}{2}-1}\sum_\tau\|E_S\phi_\tau\|^p_{L^p(B_R)}\\
&\lesssim D_{2,p}(K^2)^pK^{\frac{p}{2}-1}\mathrm{E}^p_{3,p}K^{(\frac{6}{p}-2)p}\sum_\tau\|\phi_\tau\|^p_{L^p(S)}\\
&= D_{2,p}(K^2)^pK^{5-\frac{3p}{2}}\|\phi\|^p_{L^p(S)}
\end{aligned}$$

Taking p-th root on both sides, we get

$$\|\text{narrow part}\|_{L^p(B_R)}\lesssim D_{2,p}(K^2)K^{\frac{5}{p}-\frac{3}{2}}\|\phi\|_{L^p(S)}$$

which is what we want.
Broad estimate: imitate the random translation technique given in the lecture. Define $g^v(x) := g(x-v)$, and for $v_1,...,v_n\in B_{K^2}$ broad, if $\tau_1,...,\tau_n\in S(B_{K^2})$, then similar to the lecture

$$\|E_S\phi\|^p_{L^p(B_{K^2})}\lesssim K^{O(1)}\mathbb{E}_v\Big[\Big\|\prod_{j=1}^n|E_S\phi^{v_j}_{\tau_j}|^{\frac{1}{n}}\Big\|^p_{L^p(B_{K^2})}\Big].$$

This is because when restricted on a hyperplane in $\mathbb{R}^3$, $E_S\phi$ and $E_S\phi_\tau$ can be regarded as the Fourier inverse of some function on $\mathbb{R}^2$, and translation is well-behaved in the sense that they preserve Fourier support, commute with projections and preserve L^p-norms under this setting. Hence

$$\begin{aligned}
&\sum_{\substack{B_{K^2}\subset B_R\\ \text{broad}}} \int_{B_{K^2}} |E_S\phi(x)|^p\, dx \\
\lesssim & K^{O(1)}\mathbb{E}_v\Big[\sum_{\substack{B_{K^2}\subset B_R\\ \text{broad}}} \| \prod_{\substack{\tau_1,\tau_2,\tau_3\in S(B_{K^2})\\ \text{transverse}}} |E_S\phi_{\tau_j}^{v_j}|^{\frac{1}{3}}\|^p_{L^p(B_{K^2})}\Big] \\
\lesssim & K^{O(1)}\mathbb{E}_v\Big[\sum_{\tau_1,\tau_2,\tau_3}\sum_{B_{K^2}\subset B_R} \| \prod_{j=1}^{3} |E_S\phi_{\tau_j}^{v_j}|^{\frac{1}{3}}\|^p_{L^p(B_{K^2})}\Big] \\
\lesssim & K^{O(1)}\mathbb{E}_v\Big[\sum_{\tau_1,\tau_2,\tau_3} \| \prod_{j=1}^{3} |E_S\phi_{\tau_j}^{v_j}|^{\frac{1}{3}}\|^p_{L^p(B_R)}\Big]
\end{aligned}$$

For the second $\lesssim$, the power $O(1)$ maybe bigger than the last one since there can be double counting of the τ's. We can take a linear change of variables that maps τ_j to S_j which satisfy the condition of multilinear extension. As we have seen in Problem 2 and the bound of the jacobian in the definition of transversality, the difference is a factor $K^{O(1)}$. Hence,

$$\begin{aligned}
\sum_{\substack{B_{K^2}\subset B_R\\ \text{broad}}} \int_{B_{K^2}} |E_S\phi(x)|^p\, dx &\lesssim K^{O(1)}\mathbb{E}_v\Big[\sum_{\tau_1,\tau_2,\tau_3} \mathrm{ME}^p_{3,p}(R)\prod_{j=1}^{3}\|\phi_{\tau_j}\|^{\frac{p}{3}}_{L^p(S)}\Big] \\
&\lesssim K^{O(1)}\mathrm{ME}^p_{3,p}(R)\sum_{\tau_1,\tau_2,\tau_3}\prod_{j=1}^{3}\|\phi_{\tau_j}\|^{\frac{p}{3}}_{L^p(S)} \\
&\lesssim K^{O(1)}\mathrm{ME}^p_{3,p}(R)\sum_{\tau_1,\tau_2,\tau_3}\frac{\sum_{j=1}^3\|\phi_{\tau_j}\|^p_{L^p(S)}}{3} \\
&\lesssim K^{O(1)}\mathrm{ME}^p_{3,p}(R)\sum_{\tau}\|\phi_{\tau}\|^p_{L^p(S)} \\
&= K^{O(1)}\mathrm{ME}^p_{3,p}(R)\|\phi\|^p_{L^p(S)}
\end{aligned}$$

Taking p-th root on both sides, we get

$$\|\text{narrow part}\|_{L^p(B_R)} \lesssim K^{O(1)}\mathrm{ME}_{3,p}(R)\|\phi\|_{L^p(S)}.$$

Therefore,

$$\mathrm{E}_{3,p}(R) \lesssim K^{O(1)}\mathrm{ME}_{3,p}(R) + D_{2,p}(K^2)\mathrm{E}_{3,p}(\frac{R}{K})K^{\frac{5}{p}-\frac{3}{2}}.$$

For the second part, choose $K=\log R$ and do induction on scales. Suppose $\mathrm{E}_{3,p}(\frac{R}{K}) \lesssim (R/K)^\epsilon$. Multilinear restriction theorem says

$$\Big\|\prod_{j=1}^{3}|E_{S_j}\phi_j|^{\frac{1}{3}}\Big\|_{L^3(B_R)} \lesssim R^\epsilon\prod_{j=1}^{3}\|\phi_j\|^{\frac{1}{3}}_{L^2(S_j)}.$$

Since $\mathrm{diam}(S_j) \leq 1$, we also have

$$\|E_{S_j}\phi_j\|_{L^\infty(B_R)} \leq \int_{S_j} |\phi_j(\omega)| \, d\mathrm{vol}_{S_j}\omega \lesssim \|\phi_j\|_{L^\infty(S_j)}$$

so

$$\| \prod_{j=1}^{3} |E_{S_j}\phi_j|^{\frac{1}{3}} \|_{L^\infty(B_R)} \leq \prod_{j=1}^{3} \|\phi_j\|^{\frac{1}{3}}_{L^\infty(S_j)}.$$

By interpolation, we have

$$\| \prod_{j=1}^{3} |E_{S_j}\phi_j|^{\frac{1}{3}} \|_{L^p(B_R)} \leq \prod_{j=1}^{3} \|\phi_j\|^{\frac{1}{3}}_{L^p(S_j)} \qquad \forall\, 3 \leq p \leq \infty,$$

in particular, $\mathrm{ME}_{3,p}(R) \lesssim R^\epsilon$ for $p \geq 10/3$. We know that $D_{2,p}(K^) \lesssim K^\epsilon$ and $K = \log R \lesssim R^\epsilon$. Plug in everything, we have

$$\begin{aligned}
\mathrm{E}_{3,p}(R) &\lesssim K^{O(1)}R^{\epsilon'} + K^\epsilon(\frac{R}{K})^{\epsilon''}K^{\frac{5}{p}-\frac{3}{2}} \\
&\lesssim R^\epsilon + R^{\epsilon'} \lesssim R^\epsilon
\end{aligned}$$

□

Discussion. *For $p \geq 10/3$, we have*

$$\mathrm{E}_{3,p}(R) \lesssim K^{O(1)}\mathrm{ME}_{3,p}(R) + K^\epsilon \mathrm{E}_{3,p}(\frac{R}{K}).$$

It's important to treat the powers of K in the two summands, namely $O(1)$ and ϵ differently, because we need to iterate the one on the left. To see this, consider too models:

(1) $\mathrm{E}_p(R) \leq K^{100}\mathrm{ME}_p(R) \leq K^{100}R^\epsilon$*, and we can choose* $K = \log R$*;*

(2) $\mathrm{E}_p(R) \leq K^\alpha \mathrm{E}_p(R/K) \leq K^{2\alpha}\mathrm{E}_p(R/K^2) \leq \cdots \leq R^\alpha \mathrm{E}_p(1) \lesssim R^\alpha$.

Returning to our case, let $s := \log K/\log R$, i.e. $K^s = R$, then

$$\begin{aligned}
\mathrm{E}_p(R) &\leq CK^{O(1)}\mathrm{ME}_p(R) + C_\epsilon K^\epsilon \mathrm{E}_p(\frac{R}{K}) \\
&\leq CK^{O(1)}\mathrm{ME}_p(R) + C_\epsilon K^\epsilon CK^{O(1)}\mathrm{ME}(\frac{R}{K}) + C_\epsilon^2 K^{2\epsilon}\mathrm{E}_p(\frac{R}{K^2}) \\
&\leq \cdots \textit{(repeat } s \textit{ times)} \\
&\leq CK^{O(1)}\mathrm{ME}_p(R) + \cdots + C_\epsilon^s R^\epsilon \mathrm{E}_p(1).
\end{aligned}$$

Since $C_\epsilon^s = R^{\log C_\epsilon/\log K}$, the last term requires that $K \to \infty$ as $R \to \infty$, while the first term requires $K \lesssim R^\epsilon$. Clearly $K = \log R$ works.

Problem 4. Suppose that we decompose S^{n-1} into L^{-1}-caps θ. For each cap θ, suppose that T_θ is the characteristic function of a tube of length L and radius 1 in $\mathbb{R}^n$ in the direction. Let $f = \sum_\theta T_\theta$. Prove the Kakeya maximal function conjecture for dimension $n = 4$

$$\|f\|_{L^p(\mathbb{R}^n)} \lesssim L^3 \qquad \text{for } p > \frac{4+2}{4} = \frac{3}{2}.$$

Proof. Divide S^3 into larger caps τ of diameter K^{-1} where $K << R$ is still a big number (we will choose $K = \log R$ later), and write $f = \sum_\tau f_\tau$ where $f_\tau = \sum_{\theta \subset \tau} T_\theta$. There are $\sim K^{n-1}$ such τ's. Tile $\mathbb{R}^4$ using balls of radius K^2 (I choose K^2 because I want to imitate the random translation). For each B_{K^2}, define the significant set

$$S(B_{K^2}) := \{\tau : \|f_\tau\|_{L^p(B_{K^2})} \geq \frac{1}{100\#\tau}\|f\|_{L^p(B_{K^2})}\}.$$

We say a ball B_{K^2} is 3-broad if $\exists\, \tau_1, \tau_2, \tau_3 \in S(B_{K^2})$ that are 3-transverse, and 3-narrow otherwise. Another way of seeing the 3-narrowness is that there exists a 2-dimensional plane Π such that $\mathrm{nor}(\tau)$ lie in a $1/K$-neighbourhood of Π for all $\tau \in S(Q)$. Write Broad=$\cup_{3-broad} B_{K^2}$ and Narrow=$\cup_{3-narrow} B_{K^2}$. Since $\|f\|_{L^\infty(\mathbb{R}^4)} \leq L^3$, we can do interpolation if we can prove the $p = 3/2$ case.
Broad estimate: use 3-linear Kakeya.

$$\begin{aligned}
\|f\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(\text{Broad})} &\sim \sum_{B_{K^2} \text{ broad}} \int_{B_{K^2}} |\sum_{\tau \in S(B_{K^2})} f_\tau|^{\frac{3}{2}} \\
&\lesssim K^{O(1)} \sum_{B_{K^2} \text{ broad}} \mathbb{E}_v \Big[\int_{B_{K^2}} \prod_{j=1}^{3} |f_{\tau_j(B_{K^2}), v_j}|^{\frac{1}{3}\cdot\frac{3}{2}} \Big]
\end{aligned}$$

where $\tau_j(B_{K^2}) \in S(B_{K^2})$, $j = 1, 2, 3$ are 3-transverse and $v_j \in B_{K^2}$. This is similar to the random translation technique introduced in class. Here we don't even need to consider its effect on the frequency domain, which makes it much easier. So by adding the $O(1)$ if necessary, we have

$$\|f\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(\text{Broad})} \lesssim K^{O(1)} \sum_{\substack{(\tau_1,\tau_2,\tau_3) \\ \text{3-transverse}}} \mathbb{E}_v \Big[\sum_{B_{K^2}} \int_{B_{K^2}} \prod_{j=1}^{3} |f_{\tau_j, v_j}|^{\frac{1}{2}} \Big].$$

We can translate the tubes to fit them into a ball of radius $L^{O(1)}$ (I think L^4 should work) without changing the overlap and so are the random translates.

Hence,

$$\begin{aligned}
\|f\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(\text{Broad})} &\lesssim K^{O(1)} \sum_{\substack{(\tau_1,\tau_2,\tau_3)\\ \text{3-transverse}}} \mathbb{E}_v\Big[\int_{B_{L^4}} \prod_{j=1}^{3} |f_{\tau_j,v_j}|^{\frac{1}{2}}\Big] \\
&\lesssim K^{O(1)} \sum_{(\tau_1,\tau_2,\tau_3)} L^{4\epsilon}(\frac{L}{K})^{3\cdot\frac{1}{2}\cdot 3} \qquad \#\theta \subset \tau \sim (\frac{L}{K})^3 \\
&\lesssim K^{O(1)-\frac{9}{2}} L^{\epsilon+\frac{9}{2}}
\end{aligned}$$

Hence $\|f\|_{L^{\frac{3}{2}}(\text{Broad})} \lesssim K^{O(1)} L^{\epsilon+3} \lesssim L^{\epsilon+3}$ if we take $K = \log K \lesssim L^{\epsilon}$.
Narrow estimate: use induction on scale, i.e. suppose $\|g\|_{L^{\frac{3}{2}}(\mathbb{R}^4)} \lesssim (L/K)^{3+\epsilon}$ where g is the sum of characteristic functions of tubes of length L/K, radius 1 and directions form a K/L-net on S^3. Then

$$\begin{aligned}
\|f\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(\text{Narrow})} &\sim \sum_{B_{K^2} \text{ narrow}} \int_{B_{K^2}} |\sum_{\tau \in S(B_{K^2})} f_\tau|^{\frac{3}{2}} \\
&\leq \sum_{B_{K^2} \text{ narrow}} \int_{B_{K^2}} \sum_{\tau \in S(B_{K^2})} |f_\tau|^{\frac{3}{2}} |S(B_{K^2})|^{\frac{3}{2}-1}
\end{aligned}$$

by Hölder's inequality, where $|S(B_{K^2})|$ is the cardinality of this set. Since all the significant τ's lie in a $1/K$ neighbourhood of S^1, $|S(B_{K^2})| \lesssim K$. Hence

$$\begin{aligned}
\|f\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(\text{Narrow})} &\lesssim K^{\frac{1}{2}} \sum_{B_{K^2} \text{ narrow}} \int_{B_{K^2}} \sum_{\tau \in S(B_{K^2})} |f_\tau|^{\frac{3}{2}} \\
&\leq K^{\frac{1}{2}} \sum_{B_{K^2} \text{ narrow}} \int_{B_{K^2}} \sum_{\tau} |f_\tau|^{\frac{3}{2}} \\
&= K^{\frac{1}{2}} \sum_{\tau} \int_{\mathbb{R}^4} |f_\tau|^{\frac{3}{2}}
\end{aligned}$$

For each τ, we need to take a change of variables A so that a $1/L$-net of a K^{-1}-cap τ maps to a K/L-net of the entire S^3 and the length of the corresponding tubes becomes L/K. Choose a coordinate system such that the x_1-axis points to the direction of τ and the others are orthogonal to it. Let A scales $1/K$ in x_1-axis and keep the other axes unchanged. Then the length of tubes becomes L/K and the angle between each two shorter tubes are $1/K$ times the previous. So $|\det(A)| \sim K$. Thus,

$$\int_{R^4} |f_\tau|^{\frac{3}{2}} = \int_{\mathbb{R}^4} |f_\tau \circ A|^{\frac{3}{2}} |\det(A)| \sim \int_{\mathbb{R}^4} |g_\tau|^{\frac{3}{2}} K \lesssim K(\frac{L}{K})^{\frac{3}{2}(3+\epsilon)}$$

Plug it into the previous estimate,

$$\begin{aligned}
\|f\|^{\frac{3}{2}}_{L^{\frac{3}{2}}(\text{Narrow})} &\lesssim K^{\frac{1}{2}} K^3 K (\frac{L}{K})^{\frac{3}{2}(3+\epsilon)} \qquad \#\tau \sim K \\
&= L^{\frac{3}{2}(3+\epsilon)} K^{-\frac{3}{2}\epsilon} \leq L^{\frac{3}{2}(3+\epsilon)}
\end{aligned}$$

Putting the broad part and narrow part together, we have

$$\|f\|_{L^{\frac{3}{2}}(\mathbb{R}^4)} \leq \|f\|_{L^{\frac{3}{2}}(\text{Broad})} + \|f\|_{L^{\frac{3}{2}}(\text{Narrow})} \lesssim L^{3+\epsilon}$$

$\square$

Discussion. *We tile $\mathbb{R}^4$ by balls of radius r, and we can choose r as small as we like. In fact, it will be easier if we define broad and narrow points.*
In my original solution I asked why we need to use the fact that $\#\tau \in S(B) \sim K$, why the Jacobian of the linear change of variables matters, and why we can afford to have $K^O(1)$ in the broad part but have to be careful about the exponent of K in the narrow part. Larry says that it has to do with induction. Let $C_p(L) :=$best constant such that $\|f\|_{L^p} \leq CL^3$. We have an iterative inequality of the form

$$C_p(L) \leq cK^{O(1)}L^\epsilon + c'K^\alpha C_p(\frac{L}{K}).$$

We were careful to make exponent α arbitrarily small. The first term is final, but to understand the second term we need to iterate. Suppose there was only a second term:

$$C_p(L) \leq K^\alpha C_p(\frac{L}{K}) \leq K^{2\alpha} C_p(\frac{L}{K^2}) \leq \cdots \leq L^\alpha C_p(1) = L^\alpha.$$

The final question is: what is different about restriction? In other words, why the choice of radius of smaller balls matters in restriction? Paul points out that the use of local orthogonality (which occurs implicitly in decoupling at a lower dimension) requires a ball of certain radius, and we need local orthogonality to take care of cancellation. For Kakeya problem, $f_\tau \geq 0$ so there's no cancellation.