Decoupling, Problem set 3

1. Write an outline of the proof of multilinear restriction. The outline should be a few steps, mostly in words or short equations. On the one hand, the outline should be a lot shorter than the whole proof. On the other hand, you should try to include "the main ideas". You might imagine that in a few weeks, you would try to reconstruct the proof just based on this outline. What is the key information that you should record for yourself?

In the rest of this problem set, we explore the broad/narrow decomposition, and its applications.

2. A rescaling argument. In order to use the broad/narrow approach to study restriction, there is a simple rescaling argument that we will need, and we work it out in this problem.

Let E_S denote the extension operator for the surface $S \subset \mathbb{R}^n$.

$$E_S \phi(x) = \int_S e^{2\pi i \omega x} \phi(\omega) dvol_S(\omega).$$

Let $\mathrm{E}_{S,p}(R)$ be the best constant in the inequality

$$\| E_S \phi \|_{L^p(B_R)} \le C \| \phi \|_{L^p(S)}.$$

Let $\mathrm{E}_{n,p}(R)$ be the maximum of $\mathrm{E}_{S,p}(R)$ over all "well-curved" choices of S: all S with diameter 1, C^3 norm at most 10, and principal curvatures pinched between 1/10 and 10.

Suppose that S is well-curved in this sense and that $\tau \subset S$ is a K^{-1} cap. Show that

$$\| E_S \phi_\tau \|_{L^p(B_R)} \le C(n)\, \mathrm{E}_{n,p}(R/K) K^{\frac{2n}{p} - (n-1)} \| \phi_\tau \|_{L^p(S)}.$$

The idea is to change variables so that the small cap τ is transformed into a new surface S' obeying the same assumptions as the original surface. Unwinding all the Jacobians from this change of variables leads to the powers of K on the right-hand side.

You don't have to be too careful/rigorous about checking that S' has the desired geometric properties. If you like, you can just work with the case that S is the paraboloid, and then S' will also be a paraboloid.

3. We use the broad/narrow strategy to relate the restriction problem to multilinear restriction.

Let $\mathrm{ME}_{n,p}(R)$ denote the best constant so that

$$\left\| \prod_{j=1}^n |E_{S_j} \phi_j|^{1/n} \right\|_{L^p(B_R)} \le C \prod_{j=1}^n \| \phi_j \|_{L^p(S_j)},$$

whenever S_j are surfaces with diameter at most 1, C^3 norm at most 10, and normal vector almost parallel to the x_j-direction.

Show that when $n = 3$

$$\mathrm{E}_{3,p}(R) \lesssim K^{O(1)}\, \mathrm{ME}_{3,p}(R) + \mathrm{D}_{2,p}(K^2)\, \mathrm{E}_{3,p}(R/K) K^{\frac{5}{p} - \frac{3}{2}}. \qquad (*)$$

(It is not any harder to do the n-dimensional case, but the exponents are a little bit messy.)

We know from class that $\mathrm{D}_{2,p}(K^2) \lesssim K^\epsilon$ for $2 \le p \le 4$. The multilinear restriction theorem leads directly to good bounds for $\mathrm{ME}_{n,p}(R)$. Using these facts and equation $(*)$, prove that $\mathrm{E}_{3,p}(R) \lesssim R^\epsilon$ for $p \ge 10/3$.

History. Tomas-Stein proved in the 70s that in dimension $n = 3$, $\|E_S f\|_{L^4(\mathbb{R}^3)} \lesssim \|f\|_{L^2(S)}$ when S is the 2-sphere. Strichartz generalized the estimate to the paraboloid in connection with studying the Schodinger equation giving the bound $E_{3,p} \lesssim 1$ for $p \geq 4$. People were stuck on the problem for a significant time until Bourgain proved that $E_{3,p} \lesssim 1$ for $p > 3\frac{7}{8}$ in 1991 (in "Besicovitch type maximal operators and applications to Fourier analysis"). In that paper, he first made progress on the Kakeya conjecture and then showed how to transfer progress on Kakeya into progress on restriction by using wave packets and multiscale analysis. There were a sequence of small improvements by many authors until 2001 or so, when a new method introduced by Wolff and Tao proved that $E_{3,p} \lesssim 1$ for $p > 10/3$. This bound was a big improvement over the best previous result, and it stood as the best known estimate for almost a decade. The approach using broad/narrow comes from a paper by Bourgain and me around 2010. The bound of 10/3 has been improved a little (by Bourgain and then by me) but it is still nearly state of the art.

It is worth knowing that there is an ϵ-removal theorem in the subject, which says that if $E_{n,p}(R) \lesssim R^\epsilon$, then $E_{n,q}(R) \lesssim 1$ for all $q > p$ – cf. T. Tao, "The Bochner-Riesz conjecture implies the restriction conjecture". So the bound proven in this problem actually implies that $E_{3,p}(R) \lesssim 1$, matching the bound of Wolff-Tao.

4. In this problem, we use the broad-narrow decomposition to study the Kakeya problem. Suppose that we decompose S^{n-1} into L^{-1}-caps θ. For each cap θ, suppose that T_θ is the characteristic function of a tube of length L and radius 1 in $\mathbb{R}^n$ in the direction θ. One form of the Kakeya conjecture is an L^p estimate for the sum of the T_θ.

Conjecture 1. *(Kakeya maximal function conjecture) If $f = \sum_\theta T_\theta$, then*

$$\|f\|_{L^p(\mathbb{R}^n)} \lesssim L^{n-1} \text{ for } p > \frac{n}{n-1}.$$

This estimate is sharp in the example where all the tubes go through the origin. The estimate is trivial for $p = +\infty$ because the number of caps θ is $\sim L^{n-1}$. It gets harder as p gets smaller. We will focus on dimension $n = 4$ where the broad-narrow method gives an alternate proof of very nearly the best known estimate. This estimate was proven by Tom Wolff in 1995:

Theorem 1. *(Wolff) Conjecture 1 holds for $p \geq \frac{n+2}{n}$.*

For dimension $n = 4$, we will prove that Conjecture 1 holds for $p > \frac{4+2}{4} = 3/2$.

The proof uses triliear Kakeya in $\mathbb{R}^4$. We recall the statement from last problem set. Suppose that $T_{j,\theta}$ are characteristic functions of radius 1 tubes in $\mathbb{R}^n$ that are approximately parallel to the x_j-axis. Suppose that $f_j = \sum_\theta T_{j,\theta}$, and suppose that there are N_j different $T_{j,\theta}$ in the sum.

Theorem 2. *(k-linear Kakeya in $\mathbb{R}^n$) If Q_S denotes a cube of side length S in $\mathbb{R}^n$, then*

$$\int_{Q_S} \prod_{j=1}^{k} f_j^{\frac{1}{k-1}} \lesssim S^\epsilon \prod_{j=1}^{k} N_j^{\frac{1}{k-1}}.$$

To get estimates towards Conjecture 1 in dimension 4, divide S^3 into larger caps τ, and write $f = \sum_\tau f_\tau$, where $f_\tau = \sum_{\theta \subset \tau} T_\theta$. Then divide $\mathbb{R}^4$ into "3-broad" and "3-narrow" parts, where the 3-broad part is designed so that you can bound $\|f\|_{L^p(Broad)}$ using 3-linear Kakeya. You will have to bound the narrow parts using induction.

If you're interested, you can also explore what happens in other dimensions. Can you recover Wolff's bound in any other dimension? For a given dimension n, we have to choose which k to make use of in the k-linear Kakeya. What is is the most efficient choice?