18.118 DECOUPLING THEORY HW2

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Problem1

We shall start by reformulating the problem. Denote by $\delta_{S^{n-1}}$ the delta function that is evenly distributed at the $(n-1)$ dimensional unit sphere. As a temporal distribution, its action on a Schwartz function is given by

$$\delta_{S^{n-1}}(\psi) = \int_{S^{n-1}} \psi(\omega) dvol_{S^{n-1}}$$

Similarly, we can define the unevenly distributed delta function $\phi\delta_{S^{n-1}}$ where $\phi(\omega) \in \mathscr{C}^\infty(S^{n-1})$ is the weight function. As a temporal distribution, its action is defined as

$$\phi\delta_{S^{n-1}}(\psi) = \int_{S^{n-1}} \psi(\omega)\phi(\omega) dvol_{S^{n-1}}$$

The goal of study is to understand the inverse Fourier transformation of $\phi\delta$.

We need to show for certain p and each $n > 0$ we can find some continous functions $\phi_n \in \mathscr{C}^0(S^{n-1})$ such that

$$\|E\phi_n\|_{L^p} \geq n\|\phi_n\|_\infty$$

We can strengthen this goal a little bit and try to prove for some $M > 0$,

$$\|E\phi_n\|_{L^p(B_M)} \geq n\|\phi_n\|_\infty$$

In what follows, we will take $M = R$ and construct a function ϕ_R such that

$$\|E\phi_R\|_{L^p(B_R)} \geq R^{\gamma'}\|\phi_R\|_\infty$$

for some $\gamma' > 0$.

From now on, we will focus on the construction of ϕ_R.

Take $L = R^{\frac{1}{2}}$ and choose an $1/L-$separated $10/L-$net on S^{n-1}. We use $\omega_1, \cdots, \omega_m$ to record their directions. For each direction ω_i, let θ_i be a $1/L-$cap on S^{n-1} centered at ω_i. Since the net is $1/L-$separated, these caps are disjoint. For each θ_i, we choose a smooth bump function

f_{θ_i} that is 1 almost on the whole cap θ_i and vanishes outside. The function to be considered has the form

$$\phi_R = \sum_{\theta} f_{\theta_i} e^{-i\omega_i \cdot x_i}$$

Each $\hat{\phi}_\theta = f_\theta(\omega) e^{i\omega_i \cdot x_i}$ is supported at a $n-$dim cap θ_i' of size $R^{-1} \times R^{-\frac{1}{2}} \times \cdots R^{-\frac{1}{2}}$ and its inverse Fourier transform concentrates on the translated dual rectangle $\theta_i^* + x_i$. This dual rectangle has size $R \times R^{\frac{1}{2}} \times R^{\frac{1}{2}} \times \cdots R^{\frac{1}{2}}$ with direction ω_i

Now by the counterexample of Kakeya problem, we can choose the net and x_i carefully, such that

$$|\Omega| = |\cup_i (\theta_i^* + x_i)| \leq C L^\gamma R^{\frac{n}{2}} \lesssim R^{\frac{n+\gamma}{2}}$$

The second factor comes from rescaling the picture. Now to estimate $\|E\phi_R\|_{L^p(B_R)}$, we proceed as follows and keep in mind that energy is concentrated on Ω,

$$\begin{aligned} \|E\phi_R\|^p_{L^p(B_R)} &\geq \|E\phi_R\|^p_{L^p(\Omega)} \\ &\quad \text{(we assume } \Omega \subset B_R) \\ &\geq \|E\phi_R\|^p_{L^2(\Omega)} |\Omega|^{1-\frac{p}{2}} \\ &\quad \text{(Holder Inequality)} \\ &\gtrsim \|E\phi_R\|^p_{L^2(B_R)} |\Omega|^{1-\frac{p}{2}} \\ &\quad \text{(Because energy concentrates)} \end{aligned}$$

At the last step, we replace the region Ω by B_R. It's tempting to replace it by the whole space $\mathbb{R}^n$ and use Plancherel's identity. Since $\phi_R \delta_{S^{n-1}}$ is not an L^2 function at all, this attempt won't work. The lesson is even though the energy concentrates, you still get a divergent term if integrating over a non-compact region. Thus, it is only feasible to estimate over B_R.

To do this, we need a cut off function. Choose a positive bump function $\hat{\eta}$ supported on B_1 with integral $\int_{\mathbb{R}^n} \eta = 1$. We require $\hat{\eta} \equiv 1$ on $B_{1/2}$. Then its inverse Fourier transformation η is almost a bump function of height 1 on B_1. Now consider $\eta_R = \eta(\frac{\cdot}{R})$ and $\hat{\eta}_R = \frac{1}{R^n}\hat{\eta}(\frac{\cdot}{R})$. Now

$$\|E\phi_R\|_{L^2(B_R)} \sim \|\eta_R E\phi_R\|_{L^2(\mathbb{R}^n)} = \|\delta_{S^{n-1}}\phi * \hat{\eta}_R\|_{L^2(\mathbb{R}^n)}$$

We need to understand the last convolution. It is a smooth bump function convolved with a tempered distribution. The claim is that it is a smooth function supported in R^{-1} neighborhood of the sphere with height R. To be specific, we write down the expression,

$$\delta_{S^{n-1}}\phi_R * \hat{\eta}_R(\xi) = \int_{S^{n-1}} \phi_R(\omega)\hat{\eta}_R(\xi-\omega)d\omega$$

If ξ is out of $1/R$ neighborhood of the unit sphere, then the integral is simply zero. If it is within $1/2R$ neighborhood and ξ is near the support of a cap θ (then this θ is unique by separateness), then the integral is taken over $S^{n-1}\cap B_{1/R}(\xi)$ whose $(n-1)$ dimensional area is roughly $R^{-(n-1)}$. On that region, $\hat{\eta}_R$ has height R^n and ϕ_R is constant. This shows the outcome is roughly R.

Finally, to compute the L^2 norm, since the domain is of size R^{-1} while the height is roughly R, the result is $R^{1/2}$. Now combining all these together, we have,

$$\|E\phi_R\|_{L^p(B_R)} \geq R^{\frac{1}{2}}|\Omega|^{\frac{1}{p}-\frac{1}{2}} \gtrsim R^{\frac{1}{2}+(\frac{1}{p}-\frac{1}{2})(\frac{\gamma}{2}+\frac{n}{2})}$$

The condition $\gamma' = \frac{1}{2}+(\frac{1}{p}-\frac{1}{2})(\frac{\gamma}{2}+\frac{n}{2}) > 0$ gives

$$(\frac{2n}{n-1} <)\ p < \frac{2(n+\gamma)}{(n+\gamma)-2}$$

Problem2

We shall use induction on n, the dimension of ambient Euclidean space and each time try to reduce it by one until it reaches k, the number of different directions.

Suppose the directions are along the first k coordinates, say $x_1, \cdots, x_k$. Consider the projection $\Pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$, which forgets the last component. Let Q'_S be the image of Q_S under Π and that is a $(n-1)$ dimensional cube of side length S. Let $l'_{j,a}$ be the image of $l_{j,a}$ and $T'_{j,a}$ be the characteristic function of the 1-neighborhood of $l'_{j,a}$. We shall prove

$$\int_{Q_S}\prod_{j=1}^{k}(\sum_{a=1}^{N_j}T_{j,a})^{\frac{1}{k-1}} \lesssim \int_{Q'_S}\prod_{j=1}^{k}(\sum_{a=1}^{N_j}T'_{j,a})^{\frac{1}{k-1}}$$

To do this, the first step is to use Fubini's theorem and rewrite LHS as

$$\int_{Q_S}\prod_{j=1}^{k}(\sum_{a=1}^{N_j}T_{j,a})^{\frac{1}{k-1}} = \int_{Q'_S}\int_{[-S,S]}\prod_{j=1}^{k}(\sum_{a=1}^{N_j}T_{j,a})^{\frac{1}{k-1}}$$

It's enough to show,

$$\int_{[-S,S]} \Pi_{j=1}^{k}(\sum_{a=1}^{N_j} T_{j,a})^{\frac{1}{k-1}} \leq \Pi_{j=1}^{k}(\sum_{a=1}^{N_j} T'_{j,a})^{\frac{1}{k-1}}$$

for any $x' = (x'_1, \cdots, x'_{n-1}) \in \mathbb{R}^n$. From now on, we regard $T_{j,a}$ as a function on $\{x'\} \times [-S, S]$. If $T_{j,a}$ is identically zero, then $T'_{j,a}(x') = 0$. But even though it is not zero, it is a characteristic function of compact interval of length $\lesssim 2$. This is because $\{x'\} \times [-S, S]$ and $l_{j,a}$ are almost perpendicular to each other and 1-neighborhood of $l_{j,a}$ has diameter 2. Now by Holder inequality,

$$\int_{[-S,S]} \Pi_{j=1}^{k}(\sum_{a=1}^{N_j} T_{j,a})^{\frac{1}{k-1}} \leq \Pi_{j=1}^{k}(\int_{[-S,S]}(\sum_{a=1}^{N_j} T_{j,a})^{k/(k-1)})^{1/k}$$

and it suffices to show

$$\int_{[-S,S]}(\sum_{a=1}^{N_j} T_{j,a})^{k/(k-1)} \leq (\sum_{a=1}^{N_j} T'_{j,a})^{k/(k-1)}$$

WLOG, we assume each $T'_{j,a}$ on RHS is not zero, otherwise we can simply discard that term.

Again by Holder inequality, we can use L^1 and L^∞ norm to control LHS,

$$\|\sum_{a=1}^{N_j} T_{j,a}\|_{L^{k/(k-1)}([-S,S])} \leq \|\sum_{a=1}^{N_j} T_{j,a}\|_1^{(k-1)/k} \|\sum_{a=1}^{N_j} T_{j,a}\|_\infty^{1/k}$$

Since each $T_{j,a} \leq 1$, $\|\sum_{a=1}^{N_j} T_{j,a}\|_\infty \leq N_j$. Also, since $\int_{[-S,S]} T_{j,a} \leq 2$, $\|\sum_{a=1}^{N_j} T_{j,a}\|_1 \leq 2N_j$. Therefore, RHS$\lesssim N_j$. This completes the induction step. Finally, we use multi-linear Kekaya theorem in dimension k to conclude.

The last step is not very enlightening, though it is valid as a proof. The picture is along $\{x'\} \times \mathbb{R}$, these tubes may intersects at m different points and for each m, there are $N_{m,1}, \cdots, N_{m,k}$ tubes coming from each direction. Therefore, the number of intersections at i-th point is estimated as

$$N_{i,1}N_{i_2}\cdots N_{i,k}$$

The inequality is saying (a discrete version)

$$\sum_i (N_{i,1}N_{i_2}\cdots N_{i,k})^{1/(k-1)} \leq \prod_j (\sum_i N_{i,j})^{1/(k-1)}$$

The intuition is the quantity on LHS achieves its maximum when intersections happen at a single point, say, when $m = 1$. This discrete version is proven similarly: first, use Holder inequality,

$$\sum_i (N_{i,1} N_{i_2} \cdots N_{i,k})^{1/(k-1)} \leq \prod_j (\sum_i N_{i,j}^{k/(k-1)})^{1/k}$$

and for each j, we need to show

$$\sum_i N_{i,j}^{k/(k-1)} \leq (\sum_i N_{i,j})^{k/(k-1)}$$

we can simply repeat the L^1 and L^∞ argument.

Problem3

a) Let $\Sigma_1, \cdots, \Sigma_k$ be hyperplanes inside $\mathbb{R}^n$ with normal vectors parallel to x_1-axis, $\cdots$, x_k-axis respectively. For each j, choose a small rectangle θ_j of size $R^{-1} \times R^{-\alpha} \times \cdots \times R^{-\alpha}$ and a smooth bump function $\hat{f}_j$ supported on θ_j. Here, the power α is allowed to vary (this is made possible because Σ_j doesn't have curvature) and the rectangle coincide with our old friend, cap when $\alpha = \frac{1}{2}$. Then f_j concentrates on θ_j^* and is roughly a constant function on this dual rectangle. WLOG, assume $f_j \sim 1$ on θ_j^*. If $k \geq 2$, then the product $\Pi|f_j|$ concentrates on B_{R^α}. We want to make a guess about e in the expression,

$$\| \prod_{i=1}^k |f_j|^{1/k} \|_{L^{2k/(k-1)}_{avg}(B_R)} \leq R^{e+\epsilon} \prod_{i=1}^k (\|f_i\|_{L^2_{avg}(\omega_{B_R})})^{1/k}$$

Now, LHS is roughly,

$$(\frac{\int_{B_{R^\alpha}} 1}{\int_{B_R} 1})^{(k-1)/2k} = R^{-n(1-\alpha)\frac{k-1}{2k}}$$

while RHS is

$$R^{e+\epsilon} \|f_1\|_{L^2_{avg}(B_R)} = R^{e+\epsilon} (\frac{R^{(n-1)\alpha} * R}{R^n})^{\frac{1}{2}} = R^{e+\epsilon-\frac{(1-\alpha)(n-1)}{2}}$$

Therefore, e has be greater than

$$\frac{n-1}{2}(1-\alpha) - n(1-\alpha)\frac{k-1}{2k} = \frac{n-k}{2k}(1-\alpha)$$

To make it large, we put $\alpha = 0$. This is the best we can do, because each Σ_j is only part of a plane and the radius is controlled by 1.

b) Before we start, we rewrite multilinear Kakeya in a proper way. By what is proven in the last problem, if projected to $\mathbb{R}^k \times \{0\}$, we have

$$\int_{Q_S^n}\prod_{j=1}^{k}(\sum_{a=1}^{N_j}T_{j,a})^{\frac{1}{k-1}}\lesssim\int_{Q_S^k}\prod_{j=1}^{k}(\sum_{a=1}^{N_j}T_{j,a}^{(k)})^{\frac{1}{k-1}}$$

By multilinear Kakeya in k dimension, RHS is controlled by

$$\lesssim_\epsilon\prod_{j=1}^{k}N_j^{1/(k-1)}\lesssim_\epsilon\prod_{j=1}^{k}(\frac{1}{S}\int_{Q_S^n}\sum_{a=1}^{N_j}T_{j,a})^{\frac{1}{k-1}}\lesssim_\epsilon S^{-k/(k-1)}\prod_{j=1}^{k}(\int_{Q_S^n}\sum_{a=1}^{N_j}T_{j,a})^{\frac{1}{k-1}}$$

We transform both sides to the average:

$$S^n\fint_{Q_S^n}\prod_{j=1}^{k}(\sum_{a=1}^{N_j}T_{j,a})^{\frac{1}{k-1}}\lesssim_\epsilon S^{-k/(k-1)}S^{nk/(k-1)}\prod_{j=1}^{k}(\fint_{Q_S^n}\sum_{a=1}^{N_j}T_{j,a})^{\frac{1}{k-1}}$$

Finally, we get

$$\fint_{Q_S^n}\prod_{j=1}^{k}(\sum_{a=1}^{N_j}T_{j,a})^{\frac{1}{k-1}}\lesssim_\epsilon S^{(n-k)/(k-1)}\prod_{j=1}^{k}(\fint_{Q_S^n}\sum_{a=1}^{N_j}T_{j,a})^{\frac{1}{k-1}}$$

This non-trivial power of S in front of the product will explain the appearance of e. In below, we will take S to be $R^{1/2}$. To prove multilinear restriction theorem in this case, we use induction on the scale and proceed as follows,

$$
\begin{aligned}
\fint_{B_R} \prod_{i=1}^{k} |f_j|^{2/(k-1)} &\leq Average_{B_{R^{1/2}} \subset B_R} \fint_{B_{R^{\frac{1}{2}}}} \prod_{i=1}^{k} |f_j|^{2/(k-1)} \\
\text{(By induction)} &\lesssim Average_{B_{R^{1/2}}} R^{\frac{1}{2}(\epsilon+e)\frac{2k}{k-1}} \prod_{i=1}^{k} (\|f_i\|_{L^2_{avg}(\omega_{B_{R^{1/2}}})})^{2/(k-1)} \\
\text{(Local Orthogonality)} &\lesssim Average_{B_{R^{1/2}}} R^{(\epsilon+e)\frac{k}{k-1}} \prod_{i=1}^{k} (\fint_{B_{R^{1/2}}} \sum_{\theta} |f_{i,\theta}|^2)^{1/(k-1)} \\
\text{(Local Constant)} &\lesssim Average_{B_{R^{1/2}}} R^{(\epsilon+e)\frac{k}{k-1}} \fint_{B_{R^{1/2}}} \prod_{i=1}^{k} (\sum_{\theta} |f_{i,\theta}|^2)^{1/(k-1)} \\
&= R^{(\epsilon+e)\frac{k}{k-1}} \fint_{B_R} \prod_{i=1}^{k} (\sum_{\theta} |f_{i,\theta}|^2)^{1/(k-1)} \\
\text{(Multilinear Kakeya)} &\lesssim R^{(n-k)/2(k-1)} R^{(\epsilon+e)\frac{k}{k-1}} \prod_{i=1}^{k} (\fint_{B_R} \sum_{\theta} |f_{i,\theta}|^2)^{1/(k-1)} \\
\text{(Local Orthogonality)} &\sim R^{(\epsilon+e)\frac{k}{k-1}+(n-k)/2(k-1)} \prod_{j=1}^{k} \|f_j\|^{2/(k-1)}_{L_{avg}(B_R)}
\end{aligned}
$$

Therefore, the correct power should result in

$$\frac{ek}{k-1} + (n-k)/2(k-1) = \frac{2k}{k-1} e$$

This is precisely $e = \frac{n-k}{2k}$. I should say a few words about why we used Multilinear Kakeya for $R^{1/2}$, instead of R. This is because each $|f_{i,\theta}|$ is a tube of size $R \times R^{1/2} \times \cdots R^{1/2}$. But what we proved for multilinear Kakeya is for tubes of radius 1, so we need to rescale the whole picture. Since we used mean integral, which is invariant under scaling, no other powers of R will appear due to this procedure.