Decoupling, Problem set 2

We have been working on understanding the connection between Kakeya-type estimates about the overlap properties of tubes and estimates for oscillatory integrals that involve cancellation. We will explore this connection on the problem set.

1. (The connection between the restriction problem and the Kakeya problem) Let us say that a Kakeya set of tubes of length L is a set of tubes $\{T_j\}$ in $\mathbb{R}^n$ of length L and radius 1 so that the directions of the tubes form a $1/L$-separated $10/L$-net on S^{n-1}. (In particular, the number of tubes is $\sim L^n$.) One of the forms of the Kakeya conjecture says that for a Kakeya set of tubes,

$$| \cup_j T_j | \gtrsim L^{n-\epsilon}.$$

(Remark: Naively it seems very reasonable to expect $| \cup_j T_j | \gtrsim L^n$, but Besicovitch gave a counterexample in the 1920s. In this example, $| \cup_j T_j | \sim L^n / \log L$. No one has found a worse counterexample since, but the known lower bounds are far from the conjecture.)

Suppose that the Kakeya conjecture were false – i.e. there is some *gamma* $< n$ and Kakeya sets of tubes with $| \cup T_j | \leq L^\gamma$ for arbitrarily large L.

Let $E = E_{S^{n-1}}$ be the extension operator for the sphere:

$$E\phi(x) := \int_{S^{n-1}} e^{2\pi i \omega \cdot x} \phi(\omega) dvol_{S^{n-1}}(\omega).$$

Recall that Stein conjectured $\|E\phi\|_{L^p(\mathbb{R}^n)} \lesssim \|\phi\|_{L^\infty}$ for each $p > \frac{2n}{n-1}$.

If the Kakeya conjecture were false in the sense above, prove that this restriction conjecture would also be false. To do so, consider an example built of wave packets arranged using the Kakeya set. Use the Kakeya property to help understand how many wave packets go through each ball at scale $B_{R^{1/2}}$. On each ball of radius $B_{R^{1/2}}$, estimate the L^2 norm of $E\phi$ using local orthogonality, and then put everything together to estimate $\int_{B_R} |E\phi|^p$.

We studied multilinear Kakeya and multilinear restriction. In these problems, when we work in $\mathbb{R}^n$, we have n families of objects, each family almost parallel to one of the coordinate axes. What if we had fewer families?

2. Suppose that $l_{j,a}$ are lines in $\mathbb{R}^n$, and that the angle from $l_{j,a}$ to the x_j-axis is at most $\frac{1}{100n}$. Let $T_{j,a}$ be the characteristic function of the 1-neighborhood of $l_{j,a}$. Suppose that j goes from 1 up to k, for some $k \leq n$, and a goes from 1 up to N_j. Let $Q_S \subset \mathbb{R}^n$ be a cube of side length S. Prove that

$$\int_{Q_S} \prod_{j=1}^k \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{k-1}} \lesssim S^\epsilon \prod_{j=1}^k N_j^{\frac{1}{k-1}}.$$

Hint: This can be reduced to the multilinear Kakeya inequality in k dimensions in a pretty clean way. Alternatively, if you want, you could imitate the proof of multilinear Kakeya.

3. Now we look at the multilinear restriction version of the last question. Suppose that $\Sigma_j \subset \mathbb{R}^n$ are C^2 hypersurfaces with diameter at most 1 and curvature at most 1. Suppose that each normal vector to Σ_j makes an angle at most $\frac{1}{100n}$ with the x_j-axis. Suppose that $\hat{f}_j$ is supported in $N_{1/R}\Sigma_j$.

In class, we outlined the proof of the multilinear restriction inequality, which says that

$$\Big\| \prod_{j=1}^{n} |f_j|^{\frac{1}{n}} \Big\|_{L_{avg}^{\frac{2n}{n-1}}(B_R)} \lesssim R^{\epsilon} \prod_{j=1}^{n} \left(\|f_j\|_{L_{avg}^2(\omega_{B_R})} \right)^{\frac{1}{n}}.$$

Consider what happens when we take a product over only k factors. The most interesting exponent is now $\frac{2k}{k-1}$.

a.) Look at examples and try to guess the best exponent $e = e(k, n)$ so that

$$\Big\| \prod_{j=1}^{k} |f_j|^{\frac{1}{k}} \Big\|_{L_{avg}^{\frac{2k}{k-1}}(B_R)} \lesssim R^{\epsilon} R^{e} \prod_{j=1}^{k} \left(\|f_j\|_{L_{avg}^2(\omega_{B_R})} \right)^{\frac{1}{k}}.$$

Hint: the worse example occurs when Σ_j are planes.

b.) Prove the result that you guessed in part a.) by following the idea of the proof of the multilinear restriction estimate that we discussed in class, using the k-linear Kakeya estimate in problem 2 in place of multilinear Kakeya.