## 18.118 PROBLEM SET 1

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Collaborators and Sources: I worked with Ricardo Grande-Izquierdo, Jackson Hance, and Jonathan Tidor.

1. "Suppose that

$$f(x) = \sum_{j=1}^{N} a_j e^{2\pi i j^2 x}$$

is a trigonometric polynomial whose frequencies are numbers between 1 and  $N^2$ . Prove that

$$\|f\|_{L^4([0,1])} \lesssim_{\varepsilon} N^{\varepsilon} \|f\|_{L^2([0,1])}.$$
 (1)

It is sufficient to prove that

$$\|f\|_{L^4([0,1])}^4 \lesssim_{\varepsilon} N^{\varepsilon} \|f\|_{L^2([0,1])}^4.$$
(2)

Computing  $||f||^4_{L^2([0,1])}$ , we have

$$||f||_{L^{2}([0,1])}^{4} = \left(||f||_{L^{2}([0,1])}^{2}\right)^{2}$$
$$= \left(||\hat{f}||_{\ell^{2}(\mathbb{Z})}^{2}\right)^{2}$$
$$= \left(\sum_{j=1}^{N} |a_{j}|^{2}\right)^{2}.$$

Thus, to prove (2), we must show that

$$\|f\|_{L^4([0,1])}^4 \lesssim_{\varepsilon} N^{\varepsilon} \left(\sum_{j=1}^N |a_j|^2\right)^2.$$

Expanding the LHS, we have

$$\|f\|_{L^{4}([0,1])}^{4} = \int_{[0,1]} |f|^{4} = \int_{[0,1]} |f^{2}|^{2}$$

$$= \int_{[0,1]} \left| \sum_{\substack{1 \le j,k \le N \\ 1}} a_{j} a_{k} e^{2\pi i (j^{2} + k^{2})x} \right|^{2} dx$$
(3)

For each m with  $1 \le m \le 2N^2$ , let

$$S_m = \{(j,k) : 1 \le j \le N \text{ and } j^2 + k^2 = m\}.$$

Using the fact that the functions  $e^{2\pi i m x}$  are orthogonal on  $L^2([0,1])$ , we continue from (3) and use the Cauchy-Schwarz inequality to give

$$\begin{split} \|f\|_{L^{4}([0,1])}^{4} &= \int_{[0,1]} \left| \sum_{1 \le m \le 2N^{2}} e^{2\pi i m x} \left( \sum_{(j,k) \in S_{m}} a_{j} a_{k} \right) \right|^{2} dx \\ &= \sum_{1 \le m \le 2N^{2}} \int_{[0,1]} \left| e^{2\pi i m x} \sum_{(j,k) \in S_{m}} a_{j} a_{k} \right|^{2} dx \\ &= \sum_{1 \le m \le 2N^{2}} \left| \sum_{(j,k) \in S_{m}} a_{j} a_{k} \right|^{2} \\ &\le \sum_{1 \le m \le 2N^{2}} |S_{m}| \sum_{(j,k) \in S_{m}} |a_{j}|^{2} |a_{k}|^{2}. \end{split}$$

By the number theory lemma that we discussed in class,

$$|S_m| \lesssim_{\varepsilon} N^{\varepsilon}$$

for all  $\varepsilon > 0$ , so we have

$$\begin{split} \|f\|_{L^{4}([0,1])}^{4} &\lesssim N^{\varepsilon} \sum_{1 \leq m \leq 2N^{2}} \sum_{(j,k) \in S_{m}} |a_{j}|^{2} |a_{k}|^{2} \\ &= N^{\varepsilon} \sum_{1 \leq j,k \leq N} |a_{j}|^{2} |a_{k}|^{2} \\ &= N^{\varepsilon} \left( \sum_{j=1}^{N} |a_{j}|^{2} \right)^{2} \\ &= N^{\varepsilon} \|f\|_{L^{2}([0,1])}^{4}, \end{split}$$

as required.

2. "Suppose that  $\hat{f}$  is supported in [0, 1]. In class we gave the intuition that f 'is roughly locally constant on length scales smaller than 1.' Here we pursue this question further. Suppose in addition that

$$|f(x)| \leq (1+|x|)^{10}$$

Prove that if  $[x_1, x_2] \in [-1, 1]$ , then

$$|f(x_1) - f(x_2)| \lesssim |x_1 - x_2|$$
."

Let  $x_1, x_2 \in [-1, 1]$  be fixed. Let  $\eta \in \mathcal{S}(\mathbb{R})$  so that  $\eta \equiv 1$  on [0, 1]. Then  $\hat{f} = \hat{f}\eta$ , so  $f = f * \check{\eta}$ . Since  $\eta \in \mathcal{S}(\mathbb{R})$ ,  $\check{\eta}$  is also in  $\mathcal{S}(\mathbb{R})$ . For future notational convenience, let  $\varphi = \check{\eta}$ . In particular, we can write

$$|f(x_{1}) - f(x_{2})| = \left| \int f(y)\varphi(x_{1} - y) \, dy - \int f(y)\varphi(x_{2} - y) \, dy \right|$$
  

$$= \left| \int f(y) \left( \varphi(x_{1} - y) - \varphi(x_{2} - y) \right) \, dy \right|$$
  

$$\leq \int |f(y)| \left| \varphi(x_{1} - y) - \varphi(x_{2} - y) \right| \, dy$$
  

$$\leq \int (1 + |y|)^{10} \left| \varphi(x_{1} - y) - \varphi(x_{2} - y) \right| \, dy.$$
(4)

By the Mean Value Theorem, given  $y \in \mathbb{R}$ , there exists  $x_3 = x_3(y)$  between  $x_1$  and  $x_2$  so that

$$|\varphi(x_1 - y) - \varphi(x_2 - y)| = |\varphi'(x_3 - y)||(x_1 - y) - (x_2 - y)| = |\varphi'(x_3 - y)||x_1 - x_2|.$$

Substituting this into (4), we have

$$|f(x_1) - f(x_2)| \le |x_1 - x_2| \left( \int (1 + |y|)^{10} |\varphi'(x_3 - y)| \, dy \right)$$
  
=  $|x_1 - x_2| \left( \int_{|y| \ge 1} (1 + |y|)^{10} |\varphi'(x_3 - y)| \, dy + \int_{|y| \le 1} (1 + |y|)^{10} |\varphi'(x_3 - y)| \, dy \right).$   
(5)

Since  $\varphi$  is Schwartz,  $\varphi'$  is also Schwartz, so for every M there exists  $C_M > 0$  so that

$$|\varphi'(z)| \leq C_M \left(\frac{1}{1+|z|}\right)^M.$$

for all z. In particular, there exists C > 0 so that

$$|\varphi'(z)| \le C \left(\frac{1}{1+|z|}\right)^{12}.$$
(6)

To estimate the first integral on the RHS of (5), note that since since  $x_3(y)$  is between  $x_1$  and  $x_2$  for each y, we must have

$$|x_3 - y| = |y - x_3| \ge |y| - |x_3| \ge |y| - 1.$$

Thus,

$$\left(\frac{1}{1+|x_3-y|}\right)^{12} \le \left(\frac{1}{1+(|y|-1)}\right)^{12} = \left(\frac{1}{|y|}\right)^{12}.$$

Combined with (6), this inequality implies that

$$\begin{split} \int_{|y|\geq 1} (1+|y|)^{10} |\varphi'(x_3-y)| dy &\leq C \int_{|y|\geq 1} (1+|y|)^{10} \left(\frac{1}{|y|}\right)^{12} dy \\ &\leq C \int_{|y|\geq 1} (2|y|)^{10} \left(\frac{1}{|y|}\right)^{12} \\ &= 2^{10} C \int_{|y|\geq 1} \frac{dy}{y^2}. \end{split}$$

Meanwhile, to estimate the second integral on the RHS of (5), we write

$$\int_{|y|\leq 1} (1+|y|)^{10} |\varphi'(x_3-y)| dy \leq \|\varphi'\|_{L^{\infty}(\mathbb{R})} \int_{|y|\leq 1} (1+|y|)^{10} dy \leq 2^{10} \|\varphi'\|_{L^{\infty}(\mathbb{R})}.$$

Having proved that both

$$\int_{|y|\ge 1} (1+|y|)^{10} |\varphi'(x_3-y)| dy \quad \text{and} \quad \int_{|y|\le 1} (1+|y|)^{10} |\varphi'(x_3-y)| dy$$

are finite, it follows by (5) that

$$|f(x_1) - f(x_2)| \lesssim |x_1 - x_2|$$

and that the implied constant does not depend on f. In particular, letting C be as in (6), we see that f is Lipschitz with Lipschitz constant

$$2^{10}\left(C\int_{|y|\geq 1}\frac{dy}{y^2}+\|\varphi'\|_{L^{\infty}(\mathbb{R})}\right).$$

3. "A decoupling problem. Suppose that

$$\Omega = \bigcup_{j=1}^{N} [j^2 - 1, j^2].$$

Estimate  $D_p(\Omega = \bigcup_{j=1}^N \theta_j)$  as well as you can for p in the range  $2 \le p \le \infty$ . To prove lower bounds, describe examples. To prove upper bounds, combine the argument from the first problem with tools from our second class: the local orthogonality lemma, the locally constant lemma, and the parallel decoupling lemma."

We claim that

$$D_p(\Omega = \bigcup_{j=1}^N \theta_j) \lesssim_{\varepsilon} \begin{cases} N^{\varepsilon}, & 2 \le p \le 4\\ N^{\varepsilon} N^{\frac{1}{2} - \frac{2}{p}}, & 4 \le p \le \infty. \end{cases}$$
(7)

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As a motivating example, let  $f_1$  be a bump function of height 1 which decays rapidly outside of [-1, 1] and satisfies  $f_1(0) = 1$ . For each j > 1, let

$$f_j(x) = e^{2\pi i (j^2 - 1)x} f_1(x).$$

For  $f = \sum_{j=1}^{N} f_j$ , we have that f(0) = N. Since some of the  $f_j$  oscillate more quickly than in our analogous example for  $\Omega = \bigcup_{j=1}^{N} [j-1,j]$ , we expect that there will be constructive interference on a shorter interval around 0; indeed writing

$$|f(x)| = \left| \sum_{j=1}^{N} f_j(x) \right| = \left| f_1(x) \sum_{j=1}^{N} e^{2\pi i (j^2 - 1)x} \right|$$
$$= |f_1(x)| \left| \left( 1 + \cos(3x) + \dots + \cos((N^2 - 1)x) \right) + i \left( \sin(3x) + \dots + \sin((N^2 - 1)x) \right) \right|$$

we see that to have constructive interference, it is necessary that  $(N^2 - 1)x < \pi/2$ . For a sufficient condition for constructive interference, note that if  $N^2x < \pi/4$ , then

$$|f(x)| \ge |Re(f(x))| \ge \frac{N}{\sqrt{2}}$$

Thus we have  $|f(x)| \sim N$  on an interval of width  $\sim N^{-2}$ , which implies that

$$\|f\|_{L^p} \gtrsim N \cdot N^{\frac{-2}{p}}$$

Since

$$\left(\sum_{j=1}^{N} \|f_j\|_{L^p}^2\right)^{1/2} \sim N^{\frac{1}{2}},$$

we have that

$$D_p(\Omega) \gtrsim N^{\frac{1}{2} - \frac{2}{p}}.$$

As we will soon prove, this example is sharp up to  $\varepsilon$ -loss for  $p \ge 4$ .

To prove (8), we use the following proposition, which is an analogue of our result from problem 1.

**Proposition 1.1.** Suppose that

$$supp\hat{f} \subset \Omega = \bigcup_{j=1}^{N} [j^2 - 1, j^2]$$

and that I is any interval of length 1. Then

$$\int_{I} |f|^{4} \lesssim N^{\varepsilon} ||f||^{4}_{L^{2}(\omega_{I})}$$
(8)

for some weight  $\omega_I$ .

Before proving Proposition 1.1, we will show that Proposition 1.1 implies (8). We begin by proving an analogue of the local decoupling lemma; specifically, we will show that if I is any interval of length 1,  $\omega_I$  is the weight in the proposition, and

$$f = \sum_{j} f_{j}$$

with  $\hat{f}_j$  supported in  $[j^2 - 1, j^2]$ , then

$$\|f\|_{L^{p}(I)} \lesssim \begin{cases} N^{\varepsilon} \left(\sum_{j=1}^{N} \|f_{j}\|_{L^{p}(\omega_{I})}^{2}\right)^{1/2}, & 2 \le p \le 4\\ N^{\varepsilon} N^{\frac{1}{2} - \frac{2}{p}} \left(\sum_{j=1}^{N} \|f_{j}\|_{L^{p}(\omega_{I})}^{2}\right)^{1/2}, & p \ge 4 \end{cases}$$
(9)

First, suppose that  $p \ge 4$ . In this case, given an interval I of length 1, we write

$$\|f\|_{L^{p}(I)}^{p} = \int_{I} |f|^{p} \leq \|f\|_{L^{\infty}(I)}^{p-4} \int_{I} |f|^{4} \\ \lesssim N^{\varepsilon} \|f\|_{L^{2}(\omega_{I})}^{4} \|f\|_{L^{\infty}(I)}^{p-4}.$$
(10)

By the local orthogonality lemma (proved in class),

$$\|f\|_{L^{2}(\omega_{I})}^{2} \lesssim \sum_{j} \|f_{j}\|_{L^{2}(\omega_{I})}^{2}.$$
(11)

(Note: one would actually need a statement of the local orthogonality lemma which gives an upper bound for  $||f||^2_{L^2(\omega_I)}$  rather than a bound for  $||f||^2_{L^2(I)}$ . However, adapting our proof from class to the modified statement is not difficult; all that is needed is to omit the step in which one bounds an integral over I by the integral on  $\mathbb{R}$  of  $|f|^2$  times the weight.) Meanwhile, we use the the triangle inequality, the locally constant lemma, and the Cauchy-Schwarz inequality to give

$$\begin{split} \|f\|_{L^{\infty}(I)}^{p-4} &\leq \left(\sum_{j} \|f_{j}\|_{L^{\infty}(I)}\right)^{p-4} \\ &\lesssim \left(\sum_{j=1}^{N} \|f_{j}\|_{L^{1}(\omega_{I})} \cdot 1\right)^{p-4} \\ &\leq N^{\frac{p-4}{2}} \left(\sum_{j=1}^{N} \|f_{j}\|_{L^{1}(\omega_{I})}^{2}\right)^{\frac{p-4}{2}} \end{split}$$

Since  $\mathbb{R}$  has finite total weighted measure, we have that

$$||f||_{L^1(\omega_I)} \lesssim ||f||_{L^2(\omega_I)}.$$

More generally, if  $p \leq q$ , then

$$\|\cdot\|_{L^p(\omega_I)} \lesssim \|\cdot\|_{L^q(\omega_I)} \tag{12}$$

Thus,

$$\|f\|_{L^{p}(I)}^{p-4} \lesssim N^{\frac{p-4}{2}} \left(\sum_{j=1}^{N} \|f_{j}\|_{L^{2}(\omega_{I})}^{2}\right)^{\frac{p-4}{2}}$$

Combining this result with (11), we continue from (10) to give

$$\|f\|_{L^{p}(I)}^{p} \lesssim N^{\varepsilon} N^{\frac{p-4}{2}} \left( \sum_{j=1}^{N} \|f_{j}\|_{L^{2}(\omega_{I})}^{2} \right)^{p/2}$$

$$\lesssim N^{\varepsilon} N^{\frac{p-4}{2}} \left( \sum_{j=1}^{N} \|f_{j}\|_{L^{p}(\omega_{I})}^{2} \right)^{p/2}.$$
(13)

(In the second line, we have used the fact that  $\|\cdot\|_{L^2(\omega_I)} \lesssim \|\cdot\|_{L^p(\omega_I)}$ .) Taking *p*th roots gives the  $p \ge 4$  case of (8).

We remark that this proof does not work for p < 4, because, for instance, when we said that

$$\int_{I} |f|^{p} \leq \|f\|_{L^{\infty}(I)}^{p-4} \int_{I} |f|^{4},$$

we were relying on the fact that p-4 was nonnegative to give the intermediate inequality

$$|f|^{p-4} \leq ||f||^{p-4}_{L^{\infty}(I)}$$

To prove that

$$\|f\|_{L^p(I)} \lesssim N^{\varepsilon} \left(\sum_{j=1}^N \|f_j\|_{L^p(\omega_I)}^2\right)^{1/2}$$

for  $2 \le p \le 4$ , note that if we substitute 4 for p in (13), then the first line implies that

$$\|f\|_{L^{4}(\omega_{I})} \lesssim N^{\varepsilon} \left(\sum_{j=1}^{N} \|f_{j}\|_{L^{2}(\omega_{I})}^{2}\right)^{1/2}.$$
(14)

Fixing p with  $2 \le p < 4$ , we use (12) along with (14) to give

$$\begin{split} \|f\|_{L^{2}(I)} &\lesssim \|f\|_{L^{2}(\omega_{I})} \lesssim \|f\|_{L^{4}(\omega_{I})} \lesssim N^{\varepsilon} \left(\sum_{j=1}^{N} \|f_{j}\|_{L^{2}(\omega_{I})}^{2}\right)^{1/2} \\ &\lesssim N^{\varepsilon} \left(\sum_{j=1}^{N} \|f_{j}\|_{L^{p}(\omega_{I})}^{2}\right)^{1/2}, \end{split}$$

thereby completing our proof of (9). Our claim in (8) now follows by the parallel decoupling lemma.

Having proved that Proposition 1.1 implies our claimed upper bound, we now prove Proposition 1.1.

*Proof.* (Proof of Proposition 1.1). Let  $\delta > 0$  such that  $\delta \ll 1/4$ . Let  $\eta$  be a non-negative real-valued bump function that is identically 1 on I, decays rapidly outside I, and has Fourier transform supported in  $[-\delta, \delta]$ .

We take  $\eta^2$  as a weight function to give

$$\begin{split} \|f\|_{L^{2}(\omega_{I})}^{4} &= \left(\|f\|_{L^{2}(\omega_{I})}^{2}\right)^{2} = \left(\int_{\mathbb{R}} |f|^{2} \eta^{2}\right)^{2} = \left(\int_{\mathbb{R}} \left|\hat{f} * \hat{\eta}\right|^{2}\right)^{2} \\ &= \left(\int_{\mathbb{R}} \left|\sum_{j} \left(\hat{f}_{j} * \hat{\eta}\right)\right|^{2}\right)^{2} \end{split}$$

For each j,  $\hat{f}_j * \hat{\eta}$  is supported on  $[j^2 - 1 - \delta, j^2 + \delta]$ . Since we chose  $\delta \ll 1/4 < 1/2$ , then intervals  $[j^2 - 1 - \delta, j^2 + \delta]$  are disjoint, so by orthogonality,

$$\|f\|_{L^{2}(\omega_{I})}^{4} = \left(\sum_{j} \int_{\mathbb{R}} |\hat{f}_{j} * \hat{\eta}|^{2}\right)^{2} = \left(\sum_{j} \|f_{j}\|_{L^{2}(\omega_{I})}^{2}\right)^{2}.$$
 (15)

Thus, we must show that

$$\int_{I} |f|^{4} \lesssim_{\varepsilon} N^{\varepsilon} \left( \sum_{j} \|f_{j}\|_{L^{2}(\omega_{I})}^{2} \right)^{2}.$$
(16)

Since  $\eta$  is nonnegative and is identically 1 on *I*, we have that

$$\begin{split} \int_{I} |f|^{4} &\lesssim \int_{\mathbb{R}} |f\eta|^{4} = \int_{\mathbb{R}} |f^{2}\eta^{2}|^{2} \\ &= \|f^{2}\eta^{2}\|_{L^{2}(\mathbb{R})}^{2} \\ &= \left\|\widehat{f^{2}} * \widehat{\eta^{2}}\right\|_{L^{2}(\mathbb{R})}^{2} \quad \text{(Plancherel)} \\ &= \int_{\mathbb{R}} \left|\widehat{f^{2}} * \widehat{\eta^{2}}\right|^{2} \\ &= \int_{\mathbb{R}} \left|\sum_{1 \leq j,k \leq N} (\widehat{f}_{j} * \widehat{f}_{k}) * (\widehat{\eta} * \widehat{\eta})\right|^{2} \\ &= \int_{\mathbb{R}} \left|\sum_{1 \leq j,k \leq N} (\widehat{f}_{j} * \widehat{\eta}) * (\widehat{f}_{k} * \widehat{\eta})\right|^{2}. \end{split}$$

For each pair (j,k), the convolution  $(\hat{f}_j * \hat{\eta}) * (\hat{f}_k * \hat{\eta})$  is supported on the interval  $[j^2 + k^2 - 2 - 2\delta, j^2 + k^2 + 2\delta]$ . For each  $m \ge -1$ , let  $\Omega_m$  be the interval [m, m + 1). An interval of the form  $[j^2 + k^2 - 2 - 2\delta, j^2 + k^2 + 2\delta]$  can intersect  $\Omega_m$  only if  $m \in \{j^2 + k^2 - 3, j^2 + k^2 - 2, j^2 + k^2 - 1, j^2 + k^2, j^2\}$ . By the number theory lemma from problem 1, if  $m \le 2N^2 = N^2 + N^2$ , then each of the equations

$$m = j^{2} + k^{2} - 2$$
$$m = j^{2} + k^{2} - 1$$
$$m = j^{2} + k^{2} - 1$$
$$m = j^{2} + k^{2} + 1$$

has  $\lesssim_{\gamma} (2N^2)^{\gamma}$  solutions for any  $\gamma > 0$ . Thus, for each m,

$$\#\{(j,k): [j^2+k^2-2-2\delta, j^2+k^2+2\delta] \cap \Omega_m \neq \emptyset\} \lesssim_{\varepsilon} N^{\varepsilon}$$

. 0

for any  $\varepsilon > 0$ . By Lemma 1.2, stated and proved below,

$$\int_{\mathbb{R}} \left| \sum_{1 \leq j,k \leq N} (\hat{f}_{j} * \hat{\eta}) * (\hat{f}_{k} * \hat{\eta}) \right|^{2} \lesssim_{\varepsilon} N^{\varepsilon} \sum_{1 \leq j,k \leq N} \int \left| (\hat{f}_{j} * \hat{\eta}) * (\hat{f}_{k} * \hat{\eta}) \right|^{2} \\
= N^{\varepsilon} \sum_{1 \leq j,k \leq N} \int |(f_{j}\eta)(f_{k}\eta)|^{2} \\
\lesssim N^{\varepsilon} \sum_{1 \leq j,k \leq N} \|f_{k}\eta\|_{L^{\infty}(\mathbb{R})}^{2} \int |f_{j}\eta|^{2}.$$
(18)

We claim that

$$\|f_k\eta\|_{L^{\infty}(\mathbb{R})} \lesssim \|f_k\|_{L^1(\omega_I)}.$$

To prove so, we mimic the proof of the locally constant lemma from class: Let  $\psi \in \mathcal{S}(\mathbb{R})$  satisfy  $\psi \equiv 1$  on [-2, 2], and let

$$\varphi_k(x) = \psi(x - k^2).$$

Since  $\hat{f}_k * \hat{\eta}$  is supported on  $[k^2 - 1 - \delta, k^2 + 1 + \delta]$ , we have

$$(\hat{f}_k * \hat{\eta})\varphi_k = \hat{f}_k * \hat{\eta},$$

which implies that

$$f_k\eta = (f_k\eta) * \check{\varphi_k}.$$

For any  $x \in \mathbb{R}$ , we have

$$\begin{aligned} |(f_k\eta)(x)| &= \left| \int_{\mathbb{R}} f_k(y)\eta(y)\check{\varphi_k}(x-y)\,dy \right| \\ &\leq \|\check{\varphi_k}\|_{L^{\infty}(\mathbb{R})} \int_{\mathbb{R}} |f_k(y)\eta(y)| \\ &= \|\psi\|_{L^{\infty}(\mathbb{R})} \|f_k\|_{L^1(\omega_I)}. \end{aligned}$$

Thus,

$$\|f_k\eta\|_{L^{\infty}(\mathbb{R})} \lesssim \|f_k\|_{L^1(\omega_I)},$$

and the implied constant does not depend on k. As discussed in class,  $\|\cdot\|_{L^p(\omega_I)} \lesssim \|\cdot\|_{L^p(\omega_I)}$  for  $p \leq q$ , so we have that

$$||f_k||_{L^1(\omega_I)} \lesssim ||f_k||_{L^2(\omega_I)}.$$

Continuing from (18) gives

$$\begin{split} \int_{\mathbb{R}} \left| \sum_{1 \leq j,k \leq N} (\hat{f}_j * \hat{\eta}) * (\hat{f}_k * \hat{\eta}) \right|^2 &\lesssim N^{\varepsilon} \sum_{1 \leq j,k \leq N} \|f_k \eta\|_{L^2(\omega_I)}^2 \int |f_j \eta|^2 \\ &= N^{\varepsilon} \sum_{1 \leq j,k \leq N} \|f_k \eta\|_{L^2(\omega_I)}^2 \|f_j \eta\|_{L^2(\omega_I)}^2 \\ &= N^{\varepsilon} \left( \sum_{1 \leq j \leq N} \|f_j \eta\|_{L^2(\omega_I)}^2 \right)^2. \end{split}$$

Combining this result with (17) gives (16), as required. This completes our proof of Proposition 1.1 modulo the proof of our lemma, which is provided below.  $\Box$ 

**Lemma 1.2.** Suppose that  $\Omega \subset \mathbb{R}^n$  is the disjoint union

$$\Omega = \bigsqcup_{m \in \mathbb{N}} \Omega_m,$$

and that  $g \in L^2(\Omega)$  is given by

$$g = \sum_{k=1}^{N} g_k$$

for some functions  $g_k \in L^2(\Omega)$ . For each m, let

$$S_m = \{k : 1 \le k \le N \text{ and } \operatorname{supp} g_k \cap \Omega_m \neq \emptyset\}$$

If  $|S_m| \lesssim_{\varepsilon} N^{\varepsilon}$  for all m, then

$$\int_{\Omega} |g|^2 \lesssim N^{\varepsilon} \sum_k \int_{\Omega} |g_k|^2.$$

*Proof.* Since the sets  $\Omega_m$  are disjoint, we have

$$\int |g|^2 = \sum_m \int_{\Omega_m} |g|^2 = \sum_m \int_{\Omega_m} \left| \sum_k g_k \right|^2 = \sum_m \int_{\Omega_m} \left| \sum_{k \in S_m} g_k(x) \cdot 1 \right|^2 \, dx.$$

For each x, we have by the Cauchy-Schwarz inequality that

$$\left|\sum_{k\in S_m} g_k(x) \cdot 1\right|^2 \leq |S_m| \sum_{k\in S_m} |g_k(x)|^2 \lesssim N^{\varepsilon} \sum_{k\in S_m} |g_k(x)|^2$$

Thus,

$$\int |g|^2 \lesssim N^{\varepsilon} \sum_m \sum_{k \in S_m} \int_{\Omega_m} |g_k(x)|^2 = N^{\varepsilon} \sum_{k=1}^N \sum_m \int_{\Omega_m} |g_k|^2 = N^{\varepsilon} \sum_{k=1}^N \int_{\mathbb{R}^n} |g_k|^2.$$