**Decoupling, Problem set 1**

Introductory letter: Please write me an email introducing yourself. (My email address is larryg@mit.edu) This helps me to get to know everyone. (I'll also use it to make a class email list.) What are you mostly working on and thinking about this year? Tell me a little about your background in analysis and especially Fourier analysis. Are there any topics in the course description or related to decoupling which you are particularly interested to study?

1. Suppose that $f = \sum_{j=1}^{N} a_j e^{2\pi i j^2 x}$ is a trigonometric polynomial whose frequencies are square numbers between 1 and $N^2$. Prove that

$$\|f\|_{L^4([0,1])} \lesssim_\epsilon N^\epsilon \|f\|_{L^2([0,1])}.$$

Recall this means that for any $\epsilon > 0$, there is a constant $C_\epsilon$ so that

$$\|f\|_{L^4([0,1])} \le C_\epsilon N^\epsilon \|f\|_{L^2([0,1])}.$$

This problem is based on ideas from the first lecture, and in particular you can use the following lemma from number theory:

**Lemma 1.** *The number of integer solutions to $a_1^2 + a_2^2 = M$ is $\lesssim M^\epsilon$ (for any $\epsilon > 0$).*

2. Suppose that $\hat{f}$ is supported in $[0,1]$. In class we gave the intuition that $f$ "is roughly constant on length scales smaller than 1". Here we pursue this question further. Suppose in addition that

$$|f(x)| \le (1 + |x|)^{10}.$$

Prove that if $x_1, x_2 \in [-1,1]$, then

$$|f(x_1) - f(x_2)| \lesssim |x_1 - x_2|.$$

(You can also assume a priori that $f$ is Schwartz, but the implicit constant cannot depend on $f$. You could prove this with implicit constant $10^6$, although it's not important to work out the numbers.)

3. A decoupling problem. Suppose that

$$\Omega = \cup_{j=1}^{N} [j^2 - 1, j^2].$$

Let $\theta_j = [j^2 - 1, j^2]$. Estimate $D_p(\Omega = \cup_{j=1}^{N} \theta_j)$ as well as you can for $p$ in the range $2 \le p \le \infty$. To prove lower bounds, describe examples. To prove upper bounds, combine the argument from the first problem with the tools from our second class: the local orthogonality lemma, the locally constant lemma and the parallel decoupling lemma.