18.118 DECOUPLING
LECTURE 7 NOTES

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Our goal over this lecture and part of the next lecture is to prove a weaker version of decoupling for the paraboloid. Here is the setup:

- $P = \{(\omega_1, \dots, \omega_n) \in \mathbb{R}^n : \omega_n = \omega_1^2 + \cdots + \omega_{n-1}^2,\ |\omega| = 1\}$;
- $\Omega = N_{\frac{1}{R}} P$;
- $\Omega = \bigsqcup \theta$ where θ are $R^{-1/2}$-caps;
- write $D_p(R) = D_{p,n}(R) = D_p\left(\Omega = \bigsqcup \theta\right)$.

Theorem 0.1 (Bourgain)**.** *For $2 \le p \le \frac{2n}{n-1}$ and all $\epsilon > 0$, then $D_{p,n}(R) \le C(n,\epsilon) R^\epsilon$. In other words, for $f = \sum_\theta f_\theta$ with $\operatorname{supp}(\hat{f}_\theta) \subseteq \theta$,*

$$\|f\|_{L^p} \lesssim R^\epsilon \left(\sum_\theta \|f_\theta\|_{L^p}^2 \right)^{1/2}.$$

Recall the stronger version of decoupling for the paraboloid that we will eventually prove:

Theorem 0.2 (Bourgain-Demeter)**.** *For $2 \le p \le \frac{2(n+1)}{n-1}$ and all $\epsilon > 0$, then $D_{p,n}(R) \le C(n,\epsilon) R^\epsilon$.*

One hint that Theorem 0.1 might be more tractable than Theorem 0.2 is that the maximum value of p is the critical exponent in multilinear restriction. Our proof of Theorem 0.1 indeed uses multilinear restriction as an input.

This lecture is divided into 3 parts. First are the "multiscale tools"; a statement of some of the properties of the decoupling problem that makes arguments at different scales work well. Next is a "multilinear decoupling" theorem which follows immediately from multilinear restriction. Finally is the proof of the decoupling theorem from the multilinear decoupling theorem.

1. Multiscale tools

The way decoupling is written down makes working at multiple scales especially convenient. This is made specific by the next two theorems.

Lemma 1.1. *If $L\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear change of variables, for any decoupling problem $\Omega = \bigsqcup \theta$,*

$$D_p\left(\Omega = \bigsqcup \theta\right) = D_p\left(L\Omega = \bigsqcup L\theta\right).$$

Proof. All we have to do is write the right transformation $f \mapsto \tilde{f}$ where $\operatorname{supp}(\hat{f}) \subseteq X$ implies that $\operatorname{supp}(\hat{\tilde{f}}) \subseteq LX$ and then check that the Jacobian factors that pop out cancel each other.

For any f with $\operatorname{supp}(\hat{f}) \subseteq X$, write $\tilde{f}(x) = f((L^*)^{-1}x)$. Then

$$\hat{\tilde{f}}(\omega) = \int e^{2\pi i\omega\cdot x}\tilde{f}(x)\,dx.$$

Making the change of variables $y = (L^*)^{-1}x$, this becomes

$$\int e^{2\pi i\omega\cdot L^*y} f(y)(\det L^*)\,dy = \int e^{2\pi i L\omega\cdot y} f(y)(\det L^*)\,dy = (\det L^*)\hat{f}(L\omega).$$

Furthermore, note that the same change of variables gives

$$\begin{aligned}\|\tilde{f}\|_{L^p} &= \left(\int |\tilde{f}(x)|^p\,dx\right)^{1/p}\\ &= \left(\int |f(y)|^p(\det L^*)\,dy\right)^{1/p}\\ &= (\det L^*)^{1/p}\|f\|_{L^p}.\end{aligned}$$

Then taking f with $\operatorname{supp}(\hat{f}) \subseteq \Omega$ and writing $f = \sum_\theta f_\theta$ where $\operatorname{supp}(\hat{f}_\theta) \subseteq \theta$, it follows that $\tilde{f} = \sum_\theta \tilde{f}_\theta$ where $\operatorname{supp}(\hat{\tilde{f}}) \subseteq L\Omega$ and $\operatorname{supp}(\hat{\tilde{f}}_\theta) \subseteq L\theta$.

Therefore

$$\begin{aligned}\|f\|_{L^p} &= (\det L^*)^{-1/p}\|\tilde{f}\|_{L^p}\\ &\le D_p\left(L\Omega = \bigsqcup L\theta\right)(\det L^*)^{-1/p}\left(\sum_\theta \|\tilde{f}_\theta\|_{L^p}^2\right)^{1/2}\\ &= D_p\left(L\Omega = \bigsqcup L\theta\right)\left(\sum_\theta \|f_\theta\|_{L^p}^2\right)^{1/2}.\end{aligned}$$

This proves that $D_p\left(\Omega = \bigsqcup \theta\right) \le D_p\left(L\Omega = \bigsqcup L\theta\right)$. The same argument with L replaced by L^{-1} shows the reverse inequality. $\square$

This lemma is used to show that medium-sized subsets of our paraboloid behave like the original problem.

Corollary 1.2. *Write $R = R_1 \cdot R_2$. Partition $\Omega = \bigsqcup \tau$ where the τ's are $R_1^{-1/2}$ caps. Then*

$$D_p\left(\tau = \bigsqcup_{\theta \subset \tau} \theta\right) \sim D_{p,n}(R_2).$$

Proof. Suppose τ is an $R_1^{1/2}$ cap centered at the point $\alpha = (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n)$ on the paraboloid. Consider the linear change of variables L defined by

$$L_i(\omega) = \begin{cases} R_1^{1/2}(\omega_i - \alpha_i) & \text{for } 1 \le i < n, \\ R_1\left((\omega_n - \alpha_n) - 2\sum_{j=1}^{n-1} \alpha_j(\omega_j - \alpha_j)\right) & \text{for } i = n. \end{cases}$$

L maps the paraboloid to itself, sends α to the origin, and scales up the ω_1-,…,ω_{n-1}-axes by a factor of $R_1^{1/2}$. This implies that L sends τ (approximately) to Ω. The desired statement now follows from Lemma 1.1. □

Proposition 1.3. *For $R = R_1 \cdot R_2$,*

$$D_p(R) \lesssim D_p(R_1) \cdot D_p(R_2).$$

Proof. Partition Ω into $R_1^{-1/2}$ caps τ and refine that partition into one of $R^{-1/2}$ caps θ. For a given f with $\text{supp}(\hat{f}) \subseteq \Omega$, define f_θ and f_τ in the usual way. Then

$$\begin{aligned} \|f\|_{L^p}^2 &\le D_p(R_1)^2 \sum_\tau \|f_\tau\|_{L^p}^2 \\ &\lesssim D_p(R_1)^2 \sum_\tau \left(D_p(R_2)^2 \sum_{\theta \subset \tau} \|f_\theta\|_{L^p}^2\right) \\ &\sim (D_p(R_1) D_p(R_2))^2 \sum_\theta \|f_\theta\|_{L^p}^2. \end{aligned}$$

□

Remark 1.4. *If one is not careful, it almost looks like Proposition 1.3 could prove the decoupling theorem. In particular the Proposition implies that*

$$D_{p,n}(R) \le (D_{p,n}(R^{1/2}))^2$$

(up to a small constant that can be removed if one works with a square partition instead of a partition into caps). Say that $D_{p,n}(R_0) \le C(n,\epsilon) R_0^\epsilon$ for some $R_0 > 1$, some $\epsilon > 0$, and some constant $C(n,\epsilon)$. Then it follows that $D_{p,n}(R_0^2) \le C(n,\epsilon)^2 (R_0^2)^\epsilon$ and in general that $D_{p,n}(R) \le$

$R^{\log C(n,\epsilon)/\log R_0} R^\epsilon$. This is a good bound if $C(n,\epsilon) \le 1$, but it does not say anything nearly strong enough most of the time.

However, this does imply that if we could find any fixed $R_0 > 1$ and $\epsilon > 0$ such that $D_{p,n}(R_0) \le R_0^\epsilon$, then $D_{p,n}(R) \lesssim R^\epsilon$ for all R. In principle, the decoupling theorem for any fixed ϵ can be proven by this observation together with a finite computation to check the base case. The computation would be absurdly long, so that this is not practical (even with a computer).

2. Multilinear decoupling

The setup of multilinear decoupling is the following:

- $P_1, \dots, P_n \subseteq P$ are transverse;
- $\Omega_j = N_{\frac{1}{R}} P_j$;
- $\Omega_j = \bigsqcup \theta$;
- $MD_{p,n}(R)$ is the best constant such that for $\operatorname{supp}(\hat{f}_j) \subseteq P_j$ and $f_j = \sum_\theta f_{j,\theta}$ in the usual way,

$$\left\| \prod_{j=1}^n |f_j|^{\frac{1}{n}} \right\|_{L^p} \le MD_{p,n}(R) \prod_{j=1}^n \left(\sum_{\theta \subset \Omega_j} \|f_{j,\theta}\|_{L^p}^2 \right)^{\frac{1}{2} \cdot \frac{1}{n}}.$$

It turns out that unlike in the case of restriction and multilinear restriction, decoupling and multilinear decoupling are closely related. One direction of this relation is easy.

Proposition 2.1. *$MD_{p,n}(R) \le D_{p,n}(R)$.*

Proof.

$$\left\| \prod_{j=1}^n |f_j|^{\frac{1}{n}} \right\|_{L^p} \le \prod_{j=1}^n \|f_j\|_{L^p}^{\frac{1}{n}} \le \prod_{j=1}^n \left(D_{p,n}(R) \left(\sum_{\theta \subset \Omega_j} \|f_{j,\theta}\|_{L^p}^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{n}}.$$

The first inequality follows from Hölder since

$$\| \prod |f_j|^{1/n} \|_{L^p} \le \prod \|f_j^{1/n}\|_{L^{pn}} = \prod \|f_j\|_{L^p}^{1/n}$$

and the second inequality is the definition of the decoupling constant. $\square$

By the standard argument, it is the same to define the decoupling constant as the smallest $MD_{p,n}(R)$ such that

$$\left\| \prod_{j=1}^{n} |f_j|^{\frac{1}{n}} \right\|_{L^p_{avg}(B_R)} \leq MD_{p,n}(R) \prod_{j=1}^{n} \left(\sum_{\theta \subset \Omega_j} \|f_{j,\theta}\|^2_{L^p_{avg}(\omega_{B_R})} \right)^{\frac{1}{2} \cdot \frac{1}{n}}.$$

This is convenient to use since for average-L^p norms Hölder can be written as $\|f\|_{L^p_{avg}(B_R)} \leq \|f\|_{L^q_{avg}(B_R)}$ for $p \leq q$.

Proposition 2.2. *For $2 \leq p \leq \frac{2n}{n-1}$,*

$$MD_{p,n}(R) \lesssim R^{\epsilon}.$$

Proof.

$$\begin{aligned}
&\text{(Hölder)} & \left\| \prod_{j=1}^{n} |f_j|^{\frac{1}{n}} \right\|_{L^p_{avg}(B_R)} &\leq \left\| \prod_{j=1}^{n} |f_j|^{\frac{1}{n}} \right\|_{L^{\frac{2}{n}n-1}_{avg}(B_R)} \\
&\text{(multilinear restriction)} & &\lesssim R^{\epsilon} \prod_{j=1}^{n} \|f_j\|^{\frac{1}{n}}_{L^2_{avg}(\omega_{B_R})} \\
&\text{(local orthogonality)} & &\lesssim R^{\epsilon} \prod_{j=1}^{n} \left(\sum_{\theta \subset \Omega_j} \|f_{j,\theta}\|_{L^2_{avg}(\omega_{B_R})} \right)^{\frac{1}{2} \cdot \frac{1}{n}} \\
&\text{(Hölder)} & &\leq R^{\epsilon} \prod_{j=1}^{n} \left(\sum_{\theta \subset \Omega_j} \|f_{j,\theta}\|_{L^p_{avg}(\omega_{B_R})} \right)^{\frac{1}{2} \cdot \frac{1}{n}}
\end{aligned}$$

□

Remark 2.3. *Can this proof be improved to cover the regime $\frac{2n}{n-1} < p \leq \frac{2(n+1)}{n-1}$? There is room for improvement in this argument if either of the two times that Hölder is used are not sharp. Hölder is used gives $\|f_{j,\theta}\|_{L^2_{avg}(\omega_{B_R})} \leq \|f_{j,\theta}\|_{L^p_{avg}(\omega_{B_R})}$ in the last line of the proof. This inequality is sharp when $|f_{j,\theta}|$ is close to evenly distributed over all of the ball B_R. When $|f_{j,\theta}|$ is far from evenly distributed over the ball B_R, we can improve over Holder. Later we will look at some more examples and discuss why it is also possible to get an improvement when $|f_{j,\theta}|$ is evenly distributed over B_R.*

3. Proof of the decoupling theorem

The bounds on multilinear decoupling in the previous section give bounds on the regular decoupling problem using the ideas of broad and narrow balls introduced in Lecture 6.

Lemma 3.1 (Main Lemma)**.** *For any $K \geq 1$,*

$$D_{p,n}(R) \lesssim K^{O(1)} M D_{p,n}(R) + D_{p,n-1}(K^2) D_{p,n}(R/K^2).$$

For $n = 2$ this inequality says that

$$D_{p,2}(R) \lesssim K^{O(1)} M D_{p,2}(R) + D_{p,2}(R/K^2).$$

Using the Main Lemma with $K \sim \log R$ implies Theorem 0.1 by induction on dimension and on scales.

Let $\Omega = \bigsqcup \tau$ for τ a K^{-1}-cap in P. Write $f = \sum_\tau f_\tau$ in the usual way. For $B = B_r(x_0)$ a ball of radius $r \leq R$, define the set of significant τ's to be

$$S(B) = \left\{ \tau : \|f_\tau\|_{L^p(B)} \geq \frac{1}{100 \cdot \#\tau} \|f\|_{L^p(B)} \right\}.$$

Define

$$f_B = \sum_{\tau \in S(B)} f_\tau.$$

Note that

$$0.99 \|f\|_{L^p(B)} \leq \|f_B\|_{L^p(B)} \leq 1.01 \|f\|_{L^p(B)}$$

by the triangle inequality. Define B to be **broad** if there exists $\tau_1, \ldots, \tau_n \in S(B)$ that are transverse. Define B to be **narrow** otherwise. Writing our large ball B_R as a disjoint union of smaller balls B_r, let

$$\text{Broad} = \bigsqcup_{B_r \text{ broad}} B_r \qquad \text{and} \qquad \text{Narrow} = \bigsqcup_{B_r \text{ narrow}} B_r.$$

The Main Lemma is proved by computing separate estimates on the broad and narrow regions. This gives rise to the two terms in the Main Lemma.

Lemma 3.2 (Broad estimate)**.**

$$\|f\|_{L^p(\text{Broad})} \leq r^{O(1)} M D_{p,n}(R) \left(\sum_\theta \|f_\theta\|^2_{L^p(B_R)} \right)^{1/2}.$$

The broad estimate can be proved in a (fairly) straightforward manner, as was the case in multilinear restriction. There is a slight complication which will be addressed in the next lecture where the proof is presented.

Lemma 3.3 (Narrow estimate)**.** *For B narrow, taking $r = K^2$,*

$$\|f_B\|_{L^p(B)} \lesssim D_{p,n-1}(K^2) \left(\sum_{\tau \in S(B)} \|f_\tau\|^2_{L^p(B)} \right)^{1/2}.$$

Most of the proof of the narrow estimate is presented in the next section.

Proof of Main Lemma. The narrow estimate implies that for each narrow B,

$$\|f\|_{L^p(B)} \lesssim D_{p,n-1}(K^2) \left(\sum_\tau \|f_\tau\|^2_{L^p(B)} \right)^{1/2}.$$

Parallel decoupling (Lemma 1.14 in Lecture 2) implies that the bounds on the decoupling constant of each narrow B can be combined to give a bound on the decoupling constant of all of Narrow. In particular,

$$\|f\|_{L^p(\text{Narrow})} \lesssim D_{p,n-1}(K^2) \left(\sum_\tau \|f_\tau\|^2_{L^p(B_R)} \right)^{1/2}.$$

The narrow estimate gives good decoupling that goes part of the way, from scale Ω to scale τ. Applying Proposition 1.3 turns this estimate into pretty good decoupling all the way from scale Ω to scale θ. Namely,

$$D_{p,n-1}(K^2) \left(\sum_\tau \|f_\tau\|^2_{L^p(B_R)} \right)^{1/2} \lesssim D_{p,n-1}(K^2) D_{p,n}(R/K^2) \left(\sum_\theta \|f_\theta\|^2_{L^p(B_R)} \right)^{1/2}.$$

Combining this with the broad estimate for $r = K^2$ gives

$$\|f\|_{L^p(B_R)} \lesssim \left(K^{O(1)} M D_{p,n}(R) + D_{p,n-1}(K^2) D_{p,n}(R/K^2)\right) \left(\sum_\theta \|f_\theta\|^2_{L^p(B_R)} \right)^{1/2}.$$

□

4. Proof of narrow estimate

Suppose that B is narrow. This implies that there exists a hyperplane Π^* such that all significant $\tau \in S(B)$ have normals almost in Π^*. In particular $\text{nor}(\tau)$ lies in the $O(K^{-1})$ neighborhood of Π^*.

Choose orthonormal coordinates $\eta_1, \dots, \eta_n$ such that η_n is normal to Π^*. Let $y_1, \dots, y_n$ be the dual coordinates to $\eta_1, \dots, \eta_n$. Let Π be the hyperplane $y_n = t$.

Lemma 4.1.

$$\|f_B\|_{L^p(\Pi \cap B)} \lesssim D_{p,n-1}(K^2) \left(\sum_{\tau \in S(B)} \|f_\tau\|^2_{L^p(\Pi \cap B)} \right)^{1/2}.$$

Proof. Write $(y_1, \dots, y_n) = (y', y_n)$ and $(\eta_1, \dots, \eta_n) = (\eta', \eta_n)$. Then write $g_B(y') = f_B(y', t)$ and $g_\tau(y') = f_\tau(y', t)$.

To prove this lemma it suffices to show

$$\|g_B\|_{L^p(\Pi\cap B)} \lesssim D_{p,n-1}(K^2)\left(\sum_{\tau\in S(B)} \|g_\tau\|^2_{L^p(\Pi\cap B)}\right)^{1/2}.$$

If we can describe $\operatorname{supp}(\hat{g}\tau)$, this will become a decoupling problem in $n-1$ dimensions.

$$\begin{aligned} g_\tau(y') = f_\tau(y', t) &= \int_{\mathbb{R}^n} e^{2\pi i (y',t)\cdot(\eta',\eta_n)} \hat{f}_\tau(\eta)\, d\eta \\ &= \int_{\mathbb{R}^{n-1}} e^{2\pi i y'\cdot\eta'} \left(\int_{\mathbb{R}} e^{2\pi i t\eta_n} \hat{f}_\tau(\eta)\, d\eta_n\right) d\eta'. \end{aligned}$$

Therefore the Fourier inversion formula implies

$$\hat{g}_\tau(\eta') = \int_{\mathbb{R}} e^{2\pi i t\eta_n} \hat{f}_\tau(\eta', \eta_n)\, d\eta_n.$$

In particular, $\operatorname{supp}(\hat{g}_\tau)$ is contained in the projection of τ to Π^*. Now this lemma is a consequence of the following claim. □

Claim 4.2. *For a set of τ such that each* $\operatorname{nor}(\tau)$ *lies in the $O(K^{-1})$-neighborhood of Π^*, it follows that the projections of τ onto Π^* are K^{-1}-caps in the K^{-2}-neighborhood of the $(n-1)$-dimensional paraboloid.*

Proof. Imagine $\mathbb{R}^n$ parameterized by $\eta_1, \dots, \eta_n$ with the η_n-axis coming out of the page. Imagine the paraboloid in this coordinate system, rotated by some angle. The paraboloid is divided into a bunch of K^{-1}-caps. We are interested in the caps τ whose normal points approximately in the $\eta_1, \dots, \eta_{n-1}$-plane. These are exactly the caps which lie along the outside of the projection of the paraboloid down to $(n-1)$-dimensions.

Furthermore, consider one of these K^{-1}-caps τ that lie along the edge of the paraboloid. Since the normal to τ is tangent to Π^*, the direction η_n is tangent to τ. So when projected down onto the $\eta_1, \dots, \eta_{n-1}$-plane, τ becomes a K^{-1} cap in the K^{-2}-neighborhood of the $(n-1)$-dimensional paraboloid. □

The proof of the narrow estimate follows from Lemma 4.1. Details will be presented next lecture.