18.118 DECOUPLING
LECTURE 6

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We begin by recalling basic settings of multi-linear restriction problem. Suppose $\Sigma_i, \cdots, \Sigma_n$ are some C^2 hyper-surfaces in $\mathbb{R}^n$ with diameter ≤ 1 and $|curvature| \lesssim 1$. We require their normal vectors do not vary too much: for any point $\omega \in \Sigma_j$, Angle($nor_{\Sigma_j}(\omega)$,e_j) $\leq \frac{1}{100n}$, where e_j is the unit vector parallel to x_j-axis. This also means their normal vectors are almost perpendicular to each other. If there are some functions f_j $(1 \leq j \leq n)$ whose Fourier transformations $\hat{f}_j$ are supported on the $1/R-$neighborhood of Σ_j, then theorem says,

Theorem 0.1 (Multilinear Restriction)**.**

$$\|\prod_{i=1}^{n}|f_j|^{1/n}\|_{L^{2n/(n-1)}_{avg}(B_R)} \leq R^{\epsilon}\prod_{i=1}^{n}(\|f_i\|_{L^2_{avg}(\omega_{B_R})})^{1/n} \tag{1}$$

We decompose $N_{1/R}\Sigma_j$ into disjoint union of $R^{-1/2}-$caps $\bigsqcup \theta_{j,a}$. Accordingly, each function f_j is decomposed into $\sum f_{j,\theta}$ such that $\hat{f}_{j,\theta}$ is the restriction of $\hat{f}$ on each cap $\theta \subset N_{1/R}\Sigma_j$. Then $|f_{j,\theta}|$ is roughly constant on translations of θ^* and each θ^* is a tube of size $R^{1/2} \times \cdots \times R^{1/2} \times R$. Therefore, the LHS of (1) is a quantitative measure of how these tubes of different directions and locations interfere with each other.

For the purpose of induction on scales, we also group these tubes according to their locations: let τ be a $R^{-1/4}-$cap inside $N_{R^{-1/2}}\Sigma_j$ and define

$$f_{j,\tau} = \sum_{\theta\subset\tau} f_{j,\theta}$$

Each $f_{j,\tau}$ is now locally constant on translations of the dual cap τ^* and τ^* is of a smaller size $R^{1/4} \times \cdots \times R^{1/4} \times R^{1/2}$. Geometrically, $\sum_{\theta_\tau} f_{j,\theta}$ concentrates on the overlap of translations of θ^*. Due to the size of θ^*, we can only see concentration on some $B_{R^{1/2}}$. The translations of τ^* are expected to be contained in these balls and the arrangement of these smaller tubes allows us to see how energy is actually distributed inside $B_{R^{1/2}}$.

Recall local orthogonality theorem:

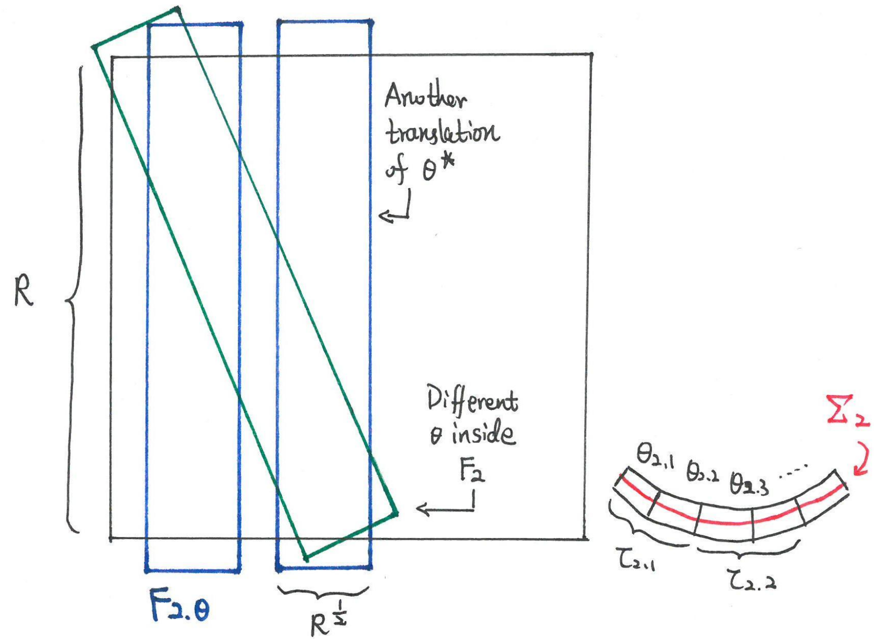

FIGURE 1. Tubes overlap.

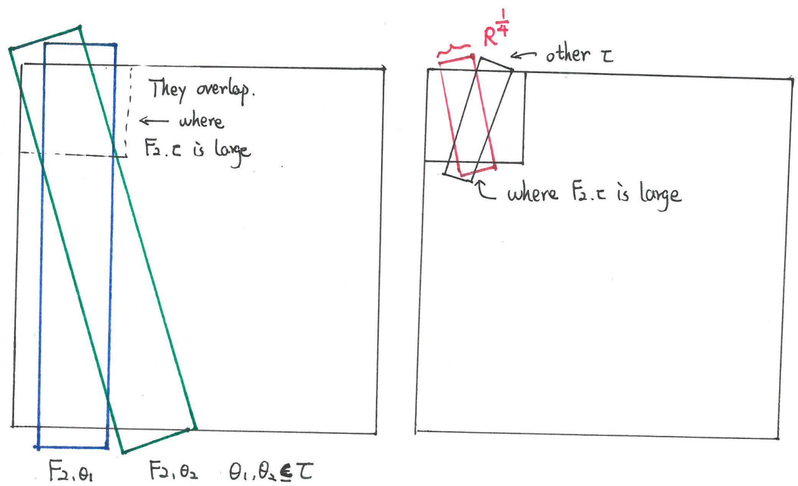

FIGURE 2. Smaller tubes refine energy distribution.

Theorem 0.2 (Local Orthogonality)**.**

$$\|f_{j,\tau}\|^2_{L^2(B_{R^{1/2}})} \lesssim \sum_{\theta\subset\tau} \|f_{j,\theta}\|^2_{L^2(\omega_{B_{R^{1/2}}})} \tag{2}$$

Remark 0.3. *The reverse inequality is not true in general. For it to be true, say,*

$$\|f_{j,\tau}\|^2_{L^2(\omega_{B_{R^{1/2}}})} \gtrsim \sum_{\theta\subset\tau} \|f_{j,\theta}\|^2_{L^2(B_{R^{1/2}})}$$

we need to use smooth cut-off functions instead of characteristic functions of caps.

The problem with our current choice is that $\hat{f}_{j,\tau}$ may have a bump at joints of two caps and this will produce some non-smoothness when passing to $\hat{f}_{j,\theta}$. The inverse Fourier transformation of the characteristic function doesn't decay fast and when we write

$$\hat{f}_{j,\theta} = \chi_\theta \hat{f}_\tau$$

and do the convolution

$$f_\theta = \check{\chi}_\theta * f_\tau$$

the energy of f_τ outside $B_{R^{1/2}}$ may contribute significantly to f_θ after convolution. This prevents the reverse inequality being true.

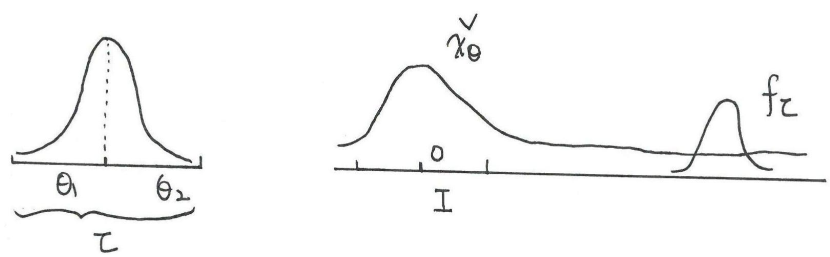

FIGURE 3. $\check{\chi}_\theta$ doesn't decay fast.

The local orthogonality theorem allows us to start with θ and keep track of

$$\sum |f_{j,\theta}(x)|^2, \sum |f_{j,\tau}(x)|^2, \sum_{\text{bigger caps}} |f_{j,\tau'}(x)|^2, \cdots$$

until $|f_j(x)|^2$. We can think of

$$|f_{j,\theta}(x)|^2 dx$$

as the energy density of $f_{j,\theta}$ and it concentrates on some beams (translations of θ^*). At each step of induction, even though the energy is conserved, beams are refined (become smaller) and they may move perpendicular to x_j-axis. Finally, $|f_j(x)|$ can be quite concentrated.

Instead of functions whose support is near Σ_j, we can study functions whose support is precisely on Σ_j:

Definition 0.4 (Extension Operator)**.** *For any smooth function $\phi \in \mathscr{C}^\infty(\Sigma)$, we define*

$$E_\Sigma \phi(x) = \int_{\Sigma_j} e^{2\pi i \omega x}\phi(x) dvol_\Sigma(\omega)$$

This operator is called the extension operator over Σ

In line with Multilinear Restriction theorem, we have

Theorem 0.5 (Multilinear Restriction')**.**

$$\| \prod_{i=1}^n |E_{\Sigma_j}\phi_j|^{1/n}\|_{L^{2n/(n-1)}(B_R)} \leq R^\epsilon \prod_{j=1}^n \|\phi_j\|^{1/n}_{L^2(\Sigma_j)}$$

This is not exactly the same form as multi-linear restriction because the integral on RHS is taken in the frequency space instead of physical space. However, the following lemma will allow us to compare them:

Lemma 0.6. *If Π is a hyperplane perpendicular to e_j (say, to x_j-axis), then*

$$\int_\Pi |E_j\phi|^2 \sim \int_{\Sigma_j} |\phi|^2$$

where E_j is E_{Σ_j} for short.

Proof. WLOG, assume $j = n$ and write a point $x \in \Pi$ as $(x_1, x_2, \cdots, x_{n-1}, t)$ where t is fixed for all $x \in \Pi$. Since Σ is normal to e_n, it is essentially a graph with $\omega_n = h(\omega') := h(\omega_1, \cdots, \omega_{n-1})$ for some function h. The volume form on Σ is

$$dvol_\Sigma = J d\omega_1\omega_2\cdots\omega_{n-1}$$

where J is the Jacobian det. Then,

$$\begin{aligned} E\phi(x) &= \int_{\mathbb{R}^{n-1}} e^{2\pi i(x_1\omega_1+\cdots+x_{n-1}\omega_{n-1})} \underbrace{e^{2\pi i t h(\omega')}\phi(\omega')J(\omega')}_{g(\omega')} d\omega' \\ &= \check{g}(x) \end{aligned}$$

Therefore,

$$\int_\Pi |E\phi|^2 = \int_\Pi |\check{g}|^2 = \int_{\mathbb{R}^{n-1}} |g|^2 = \int_{\mathbb{R}^{n-1}} J^2|\phi|^2 \sim \int_{\mathbb{R}^{n-1}} J|\phi|^2 = \int_\Sigma |\phi|^2 dvol_\Sigma$$

$\square$

In particular, as t varies, $\int_\Pi |E_j\phi|^2$ does not change. In terms of picture, this means the energy of $E_j\phi$ at each slice is roughly the same.

Corollary 0.7. *If we have (E, ϕ) as above, then*

$$\|E\phi\|_{L^2(B_R)} \lesssim R^{1/2}\|\phi\|_2$$

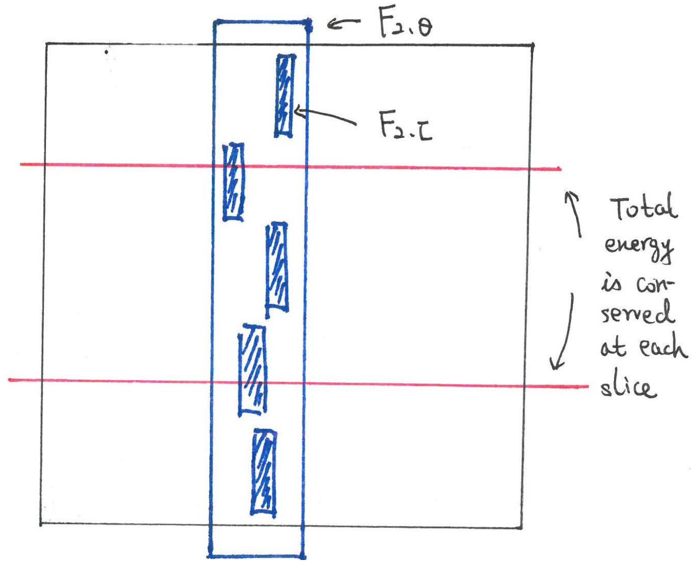

FIGURE 4. Energy at different slices is almost the same

Proof.

$$LHS^2 \leq \int_{-R}^{R} dt \int_{\Pi(t)} |E\phi|^2 \sim \int_{-R}^{R} dt \|\phi\|_2^2 \sim R\|\phi\|_2^2$$

□

Now we can compare Theorem 0.5 with our previous multi-linear restriction theorem 0.1.

After working on multi-linear restriction theorem for a while, we come back to restriction problem and see why the multi-linear version might be helpful. Suppose Σ is the unit sphere or any other smooth compact hyper-surfaces which is strictly convex in a quantitative way, say, with all principle curvatures ~ 1, then the conjecture is

Conjecture 0.8. *For $p = \frac{2n}{n-1}$,*

$$\|E\phi\|_{L^p(B_R)} \lesssim R^\epsilon \|\phi\|_{L^p(\Sigma)}$$

Remark 0.9. *On RHS, one can put $\|\phi\|_{L^q(\Sigma)}$ for $p \leq q \leq +\infty$ but not for $q < p$. In the latter case, one can produce counterexample by using a single wave packet.*

In the multi-linear case, we have n functions and their images intersect nicely. But now, we don't necessarily have this picture.

We decompose $\Sigma = \bigsqcup \tau$ into to K^{-1} caps with $K \sim log\ R$ being a parameter and let A be the set of all caps. Then $E\phi = \sum E\phi_\tau$ and

$$\begin{aligned}\int_{B_R} |E\phi|^p &= \int_{B_R} |\sum E\phi_\tau|^p \\ &= \int_{B_R} \prod_{j=1}^{n} |\sum_\tau E\phi_\tau|^{p/n} \\ &\leq K^{O(1)} \sum_{\tau_i \in A,\ 1\leq i\leq n} \int_{B_R} \prod_{i=1}^{n} |E\phi_{\tau_i}|^{p/n}\end{aligned}$$

At the last step, we used Hölder's inequality and that explains the $O(1)$ power of K on the RHS (it is related to the number of caps). We are at a situation to apply multi-linear restriction theorem if we can check the angle condition holds:

Definition 0.10. *A sequence of caps $(\tau_1, \cdots, \tau_n)$ is called transverse if there is a linear change of variables L with $|\det(L)| \lesssim K^{O(1)}$ such that $L\tau_1, \cdots, L\tau_n$ obey hypothesis of multi-linear restriction theorem* 0.5.

On S^{n-1}, $(\tau_1, \cdots, \tau_n)$ is not transverse if and only if $\tau_1, \cdots, \tau_n$ are contained in an $O(K^{-1})-$neighborhood of equator. As an application of multi-linear restriction theorem 0.5, we prove restriction theorem for $n = 2$ and for $\Sigma = S^1$.

Theorem 0.11.
$$\|E\phi\|_{L^4(B_R)} \lesssim R^\epsilon \|\phi\|_{L^4(S^1)}$$

Proof. We decompose $S^1 = \bigsqcup \tau$ into K^{-1} caps with $K \sim \log R$. The number of caps is around K. Now for $x \in B_R$, define

$$S(x) = \{\tau : |E\phi_\tau(x)| \geq \frac{1}{100K}|E\phi(x)|\}$$

The set $S(x)$ records caps that contribute non-trivially at x.

Observation.

$$\sum_{\tau \notin S(x)} |E\phi_\tau(x)| \leq \frac{1}{10}|E\phi(x)|$$

This implies

$$|\sum_{\tau \in S(x)} E\phi_\tau(x)| \sim |E\phi(x)|$$

We call x is **broad** if we can find $\tau_1, \tau_2 \in S(x)$ that (τ_1, τ_2) is transverse. We call x is **narrow** if we cannot. The board part is easy to estimate due to multi-linear restriction theorem 0.5:

$$\begin{aligned}
\int_{B_R\cap Broad}|E\phi|^4 &\lesssim (100K)^4 \sum_{(\tau_1,\tau_2),transverse}\int_{B_R}|E\phi_{\tau_1}|^2|E\phi_{\tau_2}|^2\\
\text{(Theorem 0.5)} &\lesssim R^\epsilon K^{O(1)} \sum_{(\tau_1,\tau_2),transverse}\|\phi_{\tau_1}\|_2^2\|\phi_{\tau_2}\|_2^2\\
&\lesssim R^\epsilon K^{O(1)}(\sum\|\phi_\tau\|_2^2)^2\\
\text{(Inverse local orthogonality)} &\sim R^\epsilon K^{O(1)}\|\phi\|^4_{L^2(S^1)}
\end{aligned}$$

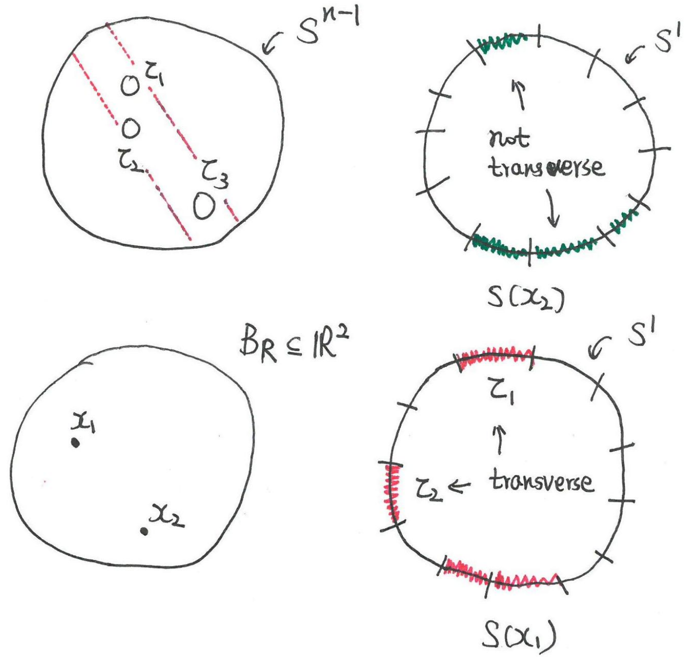

FIGURE 5. The equator, broad and narrow points

If x is narrow, since the equator is merely two points, $|S(x)| \lesssim 1$. Therefore, by Hölder's Inequality

$$|E\phi(x)|^4 \sim |\sum_{\tau\in S(x)} E\phi_\tau(x)|^4 \leq \sum_{\tau\in S(x)}|E\phi_\tau(x)|^4 \tag{3}$$

because there are not many terms in the summation. By integrating (3) over the narrow part, we get

$$\int_{B_R\cap Narrow}|E\phi|^4 \lesssim \sum_\tau \int_{B_R}|E\phi_\tau|^4$$

Remark 0.12. *This inequality trivially holds if supp$E\phi_\tau$ are disjoint.*

The remainder of the proof is based on induction on scales. By a change of variables, we transform τ into the whole circle. In light of decoupling, we define

Definition 0.13. *We define $C(R)$ to be the smallest number such that for any surface Σ that satisfies the condition in Conjecture* 0.8 *and for any smooth function $\phi \in \mathscr{C}^\infty(\Sigma)$, the inequality*

$$\|E\phi\|_{L^4(B_R)} \leq C(R)\|\phi\|_{L^4(\Sigma)}$$

holds.

In terms of this definition, the goal of Theorem 0.11 is to show $C(R) \leq C_1 R^\epsilon$ for some $C_1 > 0$.

Remark 0.14. *$C(R)$ is finite for all $R > 0$. In fact, it is easy to show $C(R) \lesssim R^{O(1)}$.*

Proof of remark.

$$\int_{B_R} |E\phi|^4 \leq |B_R|\|E\phi\|_\infty^4 \leq |B_R|\|\phi\|_1^4 \leq |B_R|\|\phi\|_4^4$$

At the last step, we used Hölder's inequality and the fact that the size of Σ is controlled by a constant. □

Lemma 0.15.

$$\|E\phi_\tau\|_{L^4(B_R)} \lesssim C(\frac{R}{K})\|\phi_\tau\|_4$$

Sketch of proof. Each τ is of size $K^{-1} \times K^{-2}$. By a linear change of variables, we stretch these two direction respectively by K and K^2 and get a new cap τ' of size 1×1. Now, $\|E\phi_\tau\|_{L^4(B_R)}$ is related to $\|E\phi_{\tilde{\tau}}\|_{L^4(B_{R/K})}$. By the definition of function $C(R)$,

$$\|E\phi_{\tilde{\tau}}\|_{L^4(B_{R/K})} \leq C(\frac{R}{K})\|\phi_{\tilde{\tau}}\|_4 \leq \|\phi_\tau\|_4$$

□

One may wonder why we need a general linear change of variables, instead of just rotations and dilation that preserve angles. The reason is that dilation dilutes the curvature. In the assumption of conjecture 0.8, curvature is needed to be bounded below.

Now apply our lemma to previous estimation for the narrow part:

$$\begin{aligned}\int_{B_R \cap Narrow} |E\phi|^4 &\lesssim \sum_\tau \int_{B_R} |E\phi_\tau|^4 \\ &\lesssim C(\frac{R}{K})^4 (\sum_\tau \|\phi_\tau\|_4^4) \\ &= C(\frac{R}{K})^4 \|\phi\|_4^4\end{aligned}$$

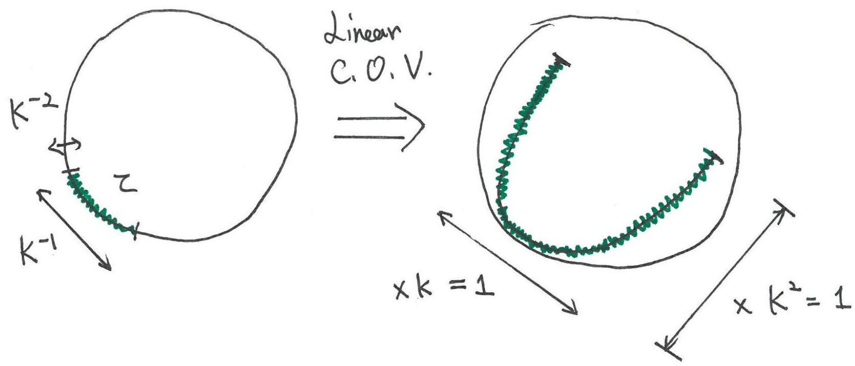

FIGURE 6. The linear change of variables.

Finally, combining narrow and broad parts of B_R, we obtain

$$C(R) \lesssim R^\epsilon K^{O(1)} + C(R/K)$$

By induction $C(R/K) \le C_1(R/K)^\epsilon$ and keep in mind that $K \sim \log(R)$. We expand the expression a little bit,

$$C(R) \le C_2 R^\epsilon + C_2 C_1 (\frac{R}{K})^\epsilon$$

No matter how large $C_1 C_2$ is, since $K \to \infty$ as $R \to \infty$, we can assume R is large enough such that $C_2 C_1 K^{-\epsilon} < 1$. We also choose $C_1 = C_2 + 1$, then

$$C(R) \le C_1 R^\epsilon$$

as desired. As for the base case of induction, we just need to start with a large R_0 and possibly choose C_1 to be bigger. □

Let's end with a few remarks for dimension 3. What will happen there? Similar to the case in $2D$, we can decompose $S^2 = \bigsqcup \theta$ into some $K^{-1}-$caps. The number of caps is now around K^2. For any $x \in B_R$, we find out all significant caps:

$$S(x) = \{\tau : |E\phi_\tau(x)| \ge \frac{1}{100K} |E\phi(x)|\}$$

We still have

$$|E\phi(x)| \sim |\sum_{\tau \in S(x)} E\phi_\tau(x)|$$

We call x is **broad** if there are $\tau_1, \tau_2, \tau_3 \in S(x)$ that (τ_1, τ_2, τ_3) is transverse. The point x is called **narrow** if else. We can still get a good bound on $\int_{B_R \cap Broad} |E\phi|^p$ by multi-linear restriction theorem.

However, for $x \in B_R$ narrow, we only know $S(x) \subset O(K^{-1})-$neighborhood of an equator. It's no longer true that $|S(x)| \lesssim 1$, which we used in an essential way in inequality (3). In 3D, $|S(x)| \lesssim K$ and Hölder's inequality will give us

$$|E\phi(x)|^p \sim |\sum_{\tau \in S(x)} E\phi_\tau(x)|^p \leq \underbrace{|S(x)|^{p-1}}_{K^{p-1}} \sum_{\tau \in S(x)} |E\phi_\tau(x)|^p$$

This little power of K will destroy everything, since in the last part of the proof for $2D$, we used $K^{-\epsilon}$ to compensate the large constant. But we don't have this profit any more.

Since $S(x)$ is supported in an S^1, it's tempting to apply our result in $2D$ and gain some control for the narrow part. But different narrow points x may give many different equators and these equators overlap. Finally, the method for $2D$ forgets cancellations near the point. Therefore, it might be helpful to define significant caps for balls of radius $R^{1/2}$ and we may gain by thinking about cancellations on that ball. That will be the task for the next lecture.