

**18.118 DECOUPLING: LECTURE 3
DECOUPLING FOR THE PARABALOID I**

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Let the parabaloid P be defined as follows:

$$P := \{\omega \in \mathbb{R}^n : \omega_n = \omega_1^2 + \dots + \omega_{n-1}^2, |\omega| \leq 1\}.$$

This is the piece of the standard parabaloid in \mathbb{R}^n which is within 1 of the origin. See figure 1. Now, let $\Omega = N_{1/R}P$ be a neighborhood of P of radius $1/R$ where R is some large parameter. This is the region which we would like to cut up and prove a decoupling result for. In particular we can cover Ω by rectangles θ so that

$$\theta \approx R^{-1/2} \times R^{-1/2} \times \dots \times R^{-1/2} \times R^{-1}$$

Let $D_p(R) := D_p(\Omega = \cup\theta)$ for this situation. Bourgain and Demeter have proved a decoupling theorem for this situation the first part of which we state below.

Theorem 0.1. *If $2 \leq p \leq p_s = 2\frac{n+1}{n-1}$, then $D_p(R) \lesssim R^\epsilon$.*

1. BUILDING BLOCKS

First we'll investigate the case of a single wavepacket. This looks like a bump function in phase space supported in one of the rectangles θ . It will turn out that we can create sharp examples using a single wavepacket from each θ . Because "typical examples" are constructed from sums of (translations in real space of) wavepackets understanding this example is a key step in the theorem.

Example 1.1. *$\hat{\psi}_\theta$ is a smooth bump function of height 1 supported in θ .*

Choose some point ω_θ at the center of θ . Then let $\theta^* = \{x \in \mathbb{R}^n \mid \omega_\theta + 1/x \in \theta\}$.

$$\theta^* \approx R^{1/2} \times R^{1/2} \times \dots \times R^{1/2} \times R^1$$

See figure 2 for the 2d case. The drawing doesn't convey this well, but θ is a very small plate while θ^* bears resemblance to a long skinny tube.

Lemma 1.2. $|\hat{\psi}_\theta| \sim |\theta|$ on θ^* and then rapidly decaying.

sketch. First we reduce to the case of $\omega_\theta = 0$.

$$\begin{aligned} |\check{\psi}_\theta(x)| &= \left| \int e^{2\pi i \omega x} \psi_\theta(\omega) d\omega \right| \\ &= \left| e^{2\pi i \omega_\theta x} \int e^{2\pi i (\omega - \omega_\theta) x} \psi_\theta(\omega) d\omega \right| \end{aligned}$$

Note that if $x \in (1/10)\theta^*$ then $|(\omega - \omega_\theta)x| < 1/10$ for all $\omega \in \theta$. Thus there can be very little cancellation so $|\check{\psi}_\theta(x)| \sim \int |\psi_\theta| \sim |\theta|$. If we're far from θ^* then we expect lots of cancellation.

Concretely: integrate by parts many times and you'll get rapid decay. \square

Remark 1.3. $\check{\psi}_\theta$ looks like the function $e^{2\pi i \omega_\theta x} |\theta| \chi_{\theta^*}$.

Using the fact that translations in real space have the same fourier support we can build more examples out of $\check{\psi}_\theta$.

Example 1.4. $\check{\psi}_\theta(x - x_0)$ and $\sum a_k \check{\psi}_\theta(x - x_k)$.

At this point it's important to recognize that examples of this type are typical.

Lemma 1.5 (Locally Constant). *Suppose $\text{Supp}(f) \subset \theta$ and T is some translation of θ^* , then*

$$\|f\|_{L^\infty(T)} \lesssim \|f\|_{L^1(\omega_T)}$$

with $\omega_T \sim 1$ on T and rapidly decaying outside T .

The proof of this lemma will, for the most part, be a transcription of the proof of the earlier locally constant lemma.

Proof. Let η be a bump function which is 1 on θ and decays rapidly outside θ . Then $\hat{f} = \eta \hat{f}$ for support reasons which implies $f = \check{\eta} * f$. Thus,

$$\|f\|_{L^\infty(T)} \leq \sup_{x \in T} \int |f(t)| |\check{\eta}(x - t)| dx \leq \int |f(t)| \sup_{x \in T} |\check{\eta}(x - t)| dx$$

As analyzed earlier $\check{\eta}(x) \sim |\theta| \sim 1/|\theta^*|$ on θ^* and decays rapidly outside this range. Thus, $\sup_{x \in T} |\check{\eta}(x - t)|$ behaves similarly on T which completes the proof. \square

Example 1.6. *Each f_θ is a single wavepacket each of which is centered at 0 all normalized so that $f_\theta(0) = 1$ (see figure 3).*

We would like to see what this example says about the value of the decoupling constant. First we can compute f at 0 using the normalization.

$$|f(0)| = \#\theta \sim R^{\frac{n-1}{2}}.$$

In a small neighborhood of 0 no significant cancellation can develop. So, for say $|x| < 1/10$

$$|f(x)| \sim R^{\frac{n-1}{2}}.$$

Using this we can bound the L^p norm of f from below,

$$\|f\|_{L^p(\mathbb{R}^n)} \geq \|f\|_{L^p(B_1)} \gtrsim R^{\frac{n-1}{2}}$$

Meanwhile our earlier analysis tells us about the L^p norm of each f_θ ,

$$\begin{aligned} \|f_\theta\|_{L^p(\mathbb{R}^n)} &\sim |\theta^*|^{1/p} \sim (R \cdot R^{\frac{n-1}{2}})^{1/p} = R^{\frac{n+1}{2p}}. \\ \left(\sum_{\theta} \|f_\theta\|_{L^p(\mathbb{R}^n)}^2\right)^{1/2} &= (\#\theta)^{1/2} R^{\frac{n+1}{2p}} = R^{\frac{n-1}{4} + \frac{n+1}{2p}}. \end{aligned}$$

From this example we see that

$$D_p(R) \gtrsim R^{\frac{n-1}{4} - \frac{n+1}{2p}}$$

The claim we'd like to prove is that up to a loss of R^ϵ this example is the worst that can happen.

Theorem 1.7 (Bourgain-Demeter).

$$D_p(R) \lesssim R^\epsilon \cdot \max(R^{\frac{n-1}{4} - \frac{n+1}{2p}}, 1).$$

It's informative to look at a plot of $\log_R(D_p(R))$ vs. p (see figure 4). From this we see that this result should be obtained from interpolation between $p = 2$, $p = \infty$ and $p = 2\frac{n+1}{n-1}$. The first two of these are easy to obtain and the line between them is the decoupling theorem we proved last time (which holds in this setting as well).

Before we move forward with outlining the ideas we need to prove the theorem we'll first investigate how a couple other examples play out.

Example 1.8. Consider the sum of N disjoint copies of example 1.5 (see figure 5).

If we let g represent example 1.7 and f represent example 1.5 then because of the rapid decay outside θ^* property we can see that

$$\|g\|_{L^p(\mathbb{R}^n)} \sim N^{1/p} \|f\|_{L^p(\mathbb{R}^n)} \text{ and } \|g_\theta\|_{L^p(\mathbb{R}^n)} \sim N^{1/p} \|f_\theta\|_{L^p(\mathbb{R}^n)}$$

From the equal scaling we can see that example 1.7 is sharp as well.

Example 1.9. Let $n = 2$ for simplicity. Consider h_θ a sum of N wavepackets in a row where $N = R^{1/2}$ (see figure 6).

From previous analysis $|h_\theta|$ looks like the indicator function on a rotated square of side length R .

Question 1.10. *How much constructive interference can occur in this situation?*

First we look at how big h can get,

$$\#\theta = R^{1/2} \quad |h| \lesssim R^{1/2}$$

Now define the red set to be the set of points such that $h(x) \sim R^{1/2}$. These are points of maximal constructive interference. In the figures such points have been marked with red throughout. Using the decoupling theorem we can put a bound on the size of the red set.

$$|Red|(R^{1/2})^6 \lesssim \|h\|_{L^6(B_R)}^6 \lesssim R^\epsilon \left(\sum \|h_\theta\|_{L^6(\omega_R)}^2 \right)^3$$

After some algebra this yields that $|Red| \lesssim R^\epsilon N$. This is quite sparse within a square of size N^4 , so our theorem will need to control constructive interference of this type quite strongly.

Example 1.11. $f = \sum_{j=1}^N e^{2\pi i(jx_1/N + j^2 x_2/N^2)}$

This example is the case of an exponential sum.

$$f(0) = N = R^{1/2}$$

This same maximum is achieved on points (Na, N^2b) for a, b integers. In a radius R ball of the origin we see $N = R^{1/2}$ points where this maximum is achieved. Using just these points we can put a lower bound on the L^p norm of f .

$$\|f\|_{L^p(B_R)} \gtrsim R$$

This is because within $1/10$ of (Na, N^2b) no significant cancellation can develop this contributes a $R^{1/2}$ and there are at least $R^{1/2}$ such points. Meanwhile a quick computation yields that the RHS of the decoupling theorem is $R^{1+1/(2p)+\epsilon}$. This says that this example has room to develop points of constructive interference that aren't obviously visible.

2. A SKETCH OF THE PROOF IDEAS

In order to get started on proving this theorem it's useful to make a table comparing the ingredients to the previous decoupling theorem we proved.

Decoupling for $\prod_{i=1}^N [i-1, i]$	Decoupling for P
Local Orthogonality	Local Orthogonality (next week)
Locally Constant	Locally Constant (above)
Tiling \mathbb{R} by $[i-1, i]$'s	Tiling of \mathbb{R}^n by translations of θ^*
$ f_\theta \sim \text{constant on tiles}$	$ f_\theta \sim \text{constant on tiles}$

In the previous theorem all the tiles were of the same type. In this case each θ has its own tiling and they have nonzero angle between them. It's this and the idea of induction on scales that we will exploit in order to prove this theorem.

If we pick a second and third scale S and T so that for example $f_\tau = \sum_{\theta \subset \tau} f_\theta$ then while the Fourier space region got bigger the tiling in real space gets smaller. Thus, while f_θ only sees a flat region f_τ will need to see disjoint regions of higher value in order for an example, f , to have too much constructive interference. However, each τ will correspond to a different arrangement of tubes and each set of tubes will have nonzero angle between them. In a certain sense it will be hard for all these tubes to "cooperate" well enough to break our decoupling theorem. See figures 7 and 8 for an illustration of this point.

Now that we're thinking about cylinders in space cooperating with each other there's a well known problem that may give us the input we need—the Kakeya problem.

Suppose $T_j \subset \mathbb{R}^n$ is a cylinder of radius 1 and length L such that T_j point in the direction $\theta_j \in S^{n-1}$ and the θ_j form a $1/L$ net of S^{n-1} . Then $\#\theta_j \approx L^{n-1}$.

Question 2.1 (Kakeya Problem). *how large can $\|\sum_j \chi_{T_j}\|_{L^p(\mathbb{R}^n)}$ be?*

This problem is quite a bit harder than it looks. An essential aspect of that hardness is that many variants of it one could produce by changing the assumptions are false. Thus, a prospective proof must use all of the given information in a precise manner.

Example 2.2. *Let g denote the sum of characteristic functions corresponding to when all of the cylinders are centered at 0.*

With some computation one can see that $g \sim L^{n-1}$ on B_1 , $g > 1$ on B_L and in between $g \sim (L/|x|)^{n-1}$. Kakeya's max function conjecture says that this example is the worst thing that can happen up to a L^ϵ loss.

Remark 2.3. *Besicovitch constructed an example in which*

$$|\cup T_j| \sim \frac{|B^n(L)|}{\log(L)}.$$

Because of this decrease in volume one of the L^p norms must increase correspondingly, so the L^ϵ fudge factor is necessary.

This conjecture has been proven true for $n = 2$ and remains open for $n \geq 3$. Interestingly, our situation only needs the weaker multilinear Keakeya in order for us to make forward progress. We will discuss this in the coming lectures.