

18.118 DECOUPLING
LECTURE 20: SHARP DECOUPLING FOR THE
MOMENT CURVE

INSTRUCTOR: LARRY GUTH
TRANSCRIBED BY ALLEN LIU

Last lecture, we showed that the decoupling constant for the moment curve satisfies $D_{k,p}(A) \leq A^\epsilon$ for $2 \leq p \leq 4k - 2$ where, as before,

$$D_{k,p}(A) = D_p(N_{A^{-k}}(\Gamma_k) = \bigsqcup A^{-1} \text{arcs})$$

We will discuss how to extend this to the sharp range $2 \leq p \leq k(k+1)$. For simplicity we will focus on the case $k = 3$ and try to improve the upper bound on p to 12. The goal of the lecture is to introduce all the tools needed to prove the sharp estimate and to give some motivation for each tool.

First we consider some examples to help build intuition. We will use the term plank to denote an $A \times A^2 \times A^3$ box in a given direction. Let f_θ be the characteristic function of a plank passing through the origin for each A^{-1} arc θ . Then $f = \sum f_\theta$ has value A in a unit ball around the origin and thus we need

$$A \lesssim \|f\|_{L^p} \lesssim D_p(A)(A \cdot A^{\frac{12}{p}})^{\frac{1}{2}} = D_p(A)A^{\frac{1}{2}}A^{\frac{6}{p}}$$

In particular, this example implies that $p = 12$ is the best we can do if we hope to show a bound of the form $D_{k,p}(A) \leq A^\epsilon$.

This example also gives us intuition as to why our proof for the non-sharp range does not extend past $p = 4k - 2$. Note that in our proof, we only looked at the planks inside balls of radius A^2 and used the fact that the f_θ are locally constant on $A \times A^2 \times \dots \times A^2$ slabs. If we consider the example above and replace each plank with a slab, we can still make the f_θ sum to $\approx A$ on the unit ball. A similar computation shows that we can only get to $p \leq 4k - 2$ by using the locally constant property on slabs.

1. MULTILINEAR TOOLS

Seeing the example above, we are motivated to develop a version of Multilinear Kakeya for planks analogous to the one we used for slabs.

Essentially, we want a statement of the form

$$\int_{B(a^3)} \prod_{i=1}^3 g_i^{\frac{s}{3}} \lesssim \prod_{i=1}^3 \left(\int g_i \right)^{\frac{s}{3}}$$

where the g_i are roughly constant on planks almost parallel to the respective coordinate axes. Note we look at balls of size a^3 since we want to be able to see the whole plank. We show that the range of s for which the above holds is $1 \leq s \leq \frac{3}{2}$. We first show that the inequality holds in the given range and then work out some examples.

Theorem 1.1. *For $1 \leq s \leq \frac{3}{2}$, if the g_i are roughly constant on planks parallel to the respective axes then we have*

$$\int_{B(a^3)} \prod_{i=1}^3 g_i^{\frac{s}{3}} \lesssim \prod_{i=1}^3 \left(\int g_i \right)^{\frac{s}{3}}$$

Proof: We can break each plank into $a \times a \times a^3$ tubes parallel to the corresponding axis. We can now apply Multilinear Kakeya on the three sets of tubes since they are parallel to the three coordinate axes and get

$$\int_{B(a^3)} \prod_{i=1}^3 g_i^{\frac{1}{2}} \lesssim \prod_{i=1}^3 \left(\int g_i \right)^{\frac{1}{2}}$$

meaning that the inequality we want is true up to $s = \frac{3}{2}$.

We next show that the range in the above theorem is sharp. The example we build is also based on the sharp example in Multilinear Kakeya where all of the tubes intersect in a box. Indeed for a given direction, we can stack together a planks of size $a \times a^2 \times a^3$ to get a fat $a^2 \times a^2 \times a^3$ tube. Now these fat tubes can intersect in an $a^2 \times a^2 \times a^2$ box and a simple computation gives that the optimal exponent in this example is $\frac{3}{2}$.

Theorem 0.1 gives us the following estimate on the Multilinear decoupling constant for $\frac{2p}{3} \geq 2$

$$M_{p, \frac{2p}{3}}(a, a) \lesssim M_{p, \frac{2p}{3}}(a^3, a)$$

We can compare this to our estimate using slabs which is

$$M_{p, \frac{p}{3}}(a, a) \lesssim M_{p, \frac{p}{3}}(a^2, a)$$

for $\frac{p}{3} \geq 2$. Note it is not clear that our new estimate is better since although increasing r from a to a^3 is better, it requires q to be $\frac{2p}{3}$ instead of just $\frac{p}{3}$.

Furthermore, if we consider the optimal example described at the beginning (with one plank through the origin in every direction), we can show that our Multilinear Keakeya on planks is not tight. Note first that if we have one plank parallel to every axis, the intersection is an $a \times a \times a$ box and thus the range for which the inequality holds is $1 \leq s \leq 2$. The case where there is one plank in every direction can be reduced to the above case by clustering the planks around each of the axes. To keep our estimates sharp, we will instead use Multilinear Keakeya on planks only to go from $r = a^2$ to $r = a^3$. We claim

$$(MKP2) \quad M_{p, \frac{2p}{3}}(a^2, a) \lesssim M_{p, \frac{2p}{3}}(a^3, a)$$

Proof. In the above (we will just show the case $p = 3, \frac{2p}{3} = 2$), we have

$$\begin{aligned} LHS &= Avg_{B(a^3) \subset B(R)} \left[Avg_{B(a^2) \subset B(a^3)} \prod_{\theta} (\sum \|f_{i, \theta}\|_{L^2(B_{a^2})}^2)^{\frac{1}{2}} \right] \\ &\lesssim Avg_{B(a^3) \subset B(R)} \left[\int_{B_{a^3}} (\prod_{\theta} (\sum \|f_{i, \theta}\|_{L^2(B_{a^2})}^2))^{\frac{1}{2}} \right] \\ &\lesssim Avg_{B(a^3) \subset B(R)} \left[\prod_{\theta} \left(\int_{B_{a^3}} \sum \|f_{i, \theta}\|_{L^2(B_{a^2})}^2 \right)^{\frac{1}{2}} \right] \\ &\lesssim M_{p, \frac{2p}{3}}(a^3, a) \end{aligned}$$

where we used plank Keakeya and the fact that the $g_i = \sum_{\theta} \|f_{i, \theta}\|_{L^2(B_{a^2})}^2$ are roughly constant on $a^2 \times a^2 \times a^3$ boxes (since each $f_{i, \theta}$ is roughly constant on planks).

2. REDUCING TO LOWER DIMENSIONS

It turns out that we still need additional tools to prove the sharp decoupling theorem on the moment curve. In this section we present a set of tools that will allow us to use induction on dimension and complete the proof.

We will try to understand the following: let θ be A^{-1} arcs such that for all θ , $|f_\theta| \sim 1$ on a ball of radius A (denoted by B_A). What can we say about f_τ on B_A where τ are $A^{-\frac{1}{3}}$ arcs. From local orthogonality, we have an estimate of the form

$$\|f_\tau\|_{L^2(B_A)}^2 \lesssim \sum_{\theta \subset \tau} \|f_\theta\|_{L^2(B_A)}^2$$

From 2-dimensional decoupling, we have that if $N_{a^{-2}}(\Gamma_2) = \sqcup a^{-1} \text{arcs}$ then

$$(D2) \quad \|f\|_{L^p(B_{a^2})} \lesssim a^\epsilon \left(\sum_{\theta} \|f_\theta\|_{L^p(B_{a^2})}^2 \right)^{\frac{1}{2}}$$

for $2 \leq p \leq 6$. We claim that the exact same statement holds if we consider 3 dimensional arcs.

Proof. We decompose the ball B_{a^2} into planes $\Pi_h = \{x_3 = h\}$ and use two dimensional decoupling on each plane. We define the functions $g_H(x_1, x_2) = f(x_1, x_2, H)$, $g_{H,\theta}(x_1, x_2) = f_\theta(x_1, x_2, H)$ and decompose $g_H = \sum g_{H,\theta}$. Note that the Fourier support of each $g_{H,\theta}$ is contained in the projection of θ onto the xy plane and thus the two dimensional decoupling statement holds. We can then use parallel decoupling to combine the slices.

We have that

$$\|f_\tau\|_{L^6(B_A)}^2 \lesssim A^\epsilon \left(\sum_{\gamma \sim A^{-\frac{1}{2}} \text{arcs}} \|f_\gamma\|_{L^6(B_A)}^2 \right)^{\frac{1}{2}}$$

and also

$$\|f_\gamma\|_{L^2(B_A)}^2 \lesssim \sum_{\theta \subset \gamma} \|f_\theta\|_{L^2(B_A)}^2$$

3. MAIN PROOF

We now list the set of all of our tools. We have from local orthogonality and the reduction of dimension argument above,

$$(O) \quad M_{p,2}(r, a) \lesssim M_{p,2}(r, r)$$

$$(D2) \quad M_{p,6}(r, a) \lesssim M_{p,6}(r, r^{\frac{1}{2}}) \forall a \leq r^{\frac{1}{2}}$$

We also have from Multilinear Keakeya for slabs and planks

$$(MKS) \quad M_{p, \frac{p}{3}}(a, a) \lesssim M_{p, \frac{p}{3}}(a^2, a)$$

$$(MKP2) \quad M_{p, \frac{2p}{3}}(a^2, a) \lesssim a^\epsilon M_{p, \frac{2p}{3}}(a^3, a)$$

Combining these tools, we can complete the proof of the sharp decoupling estimate in a similar manner to the non-sharp one. We decompose into broad and narrow balls and use multilinear tools in the broad case and induction on dimension in the narrow case. We also use induction to handle terms of the form $M_{p,p}(r, a)$ as before. The details of the algebra are omitted here.