18.118 DECOUPLING LECTURE 19

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We resume our study of decoupling for the moment curve. We recall that the moment curve $\Gamma_k \subset \mathbb{R}^k$ is defined as the image of $[0,1]$ under the function

$$\gamma(t) = (t, t^2, \ldots, t^k)$$

and that $D_{k,p}(A)$ denotes the decoupling constant for a decomposition of the A^{-k} neighborhood of Γ_k into A^{-1} arcs. That is,

$$D_{k,p}(A) = D_p\left(N_{A^{-k}}(\Gamma_k) = \bigsqcup A^{-1}\text{arcs}\right).$$

Last time we stated the following two theorems, both due to Bourgain and Demeter.

Theorem 0.1. *For $2 \leq p \leq 4k-2$, we have that*

$$D_{k,p}(A) \lesssim A^{\varepsilon}.$$

Theorem 0.2. *For $2 \leq p \leq k(k+1)$, we have that*

$$D_{k,p}(A) \lesssim A^{\varepsilon}.$$

The exponent $k(k+1)$ in Theorem (0.2) is sharp.

Remark 0.3. *We note that for $k=2$, the upper bounds for p in the non-sharp theorem and the sharp theorem coincide with each other and also with the sharp exponent for decoupling on the paraboloid. This is because for $k=2$, the moment curve is a parabola.*

Previously, we proved a broad/narrow decomposition that reduced the proof of the weak decoupling theorem to a multilinear estimate. Letting $K \ll A$, we wrote

$$f = \sum_{\tau} f_\tau = \sum_{\tau} \sum_{\theta \subset \tau} f_{\tau,\theta},$$

where each τ was a K^{-1} arc of the moment curve and each θ was an A^{-1} arc. In the broad case, we claimed that if $\tau_1, \ldots, \tau_k$ were transverse K^{-1} arcs and $f_i = f_{\tau_i}$, then

$$(1) \qquad \fint \prod_{i=1}^{k} |f_i|^{\frac{1}{k}\cdot p} \lesssim A^{\varepsilon} \left(\sum_{\theta\ A^{-1}\text{-arcs}} \|f_\theta\|^2_{L^p(B_R)} \right)^{\frac{1}{2}\cdot p}.$$

(Strictly speaking, the RHS should actually involve norms with respect to some weight ω_{B_R}, but we will not belabor such details.)

Today we will prove this claim. To encapsulate some of the main concepts in our proof, let

$$M_{p,q}(r,a) := \underset{B_r \subset B_R}{Avg} \prod_{i=1}^{k} \left(\sum_{\theta \ a^{-1}\text{-arcs}} \|f_{i,\theta}\|^2_{L^p(B_r)} \right)^{\frac{1}{2}\cdot\frac{1}{k}\cdot p}.$$

The expressions $M_{p,q}$ satisfy the following properties:

(a) Orthogonality (O): If $a \le r$, then

$$M_{p,2}(r,a) \le M_{p,2}(r,r).$$

(b) Multilinear Kakeya for Slabs, version 1 (MKS1):

$$M_{2k,2}(a,a) \lesssim M_{2k,2}(a^2,a).$$

(c) Multilinear Kakeya for Slabs, version 2 (MKS2): If $p \ge 2k$, then

$$M_{p,\frac{p}{k}}(a,a) \lesssim M_{p,\frac{p}{k}}(a^2,a).$$

(d) $M_{p,q}(\cdot,\cdot)$ is increasing in q.

(e) Hölder (H): If $\frac{1}{q} = \frac{\alpha}{q_1} + \frac{1-\alpha}{q_2}$, then

$$\|\cdot\|_q \le \|\cdot\|_{q_1}^{\alpha} \|\cdot\|_{q_2}^{1-\alpha},$$

which implies that

$$M_{p,q} \le M_{p,q_1}^{\alpha} M_{p,q_2}^{1-\alpha}.$$

To prove (1), we begin by observing that left-hand side satisfies

$$⨍ \prod_{i=1}^{k} |f_i|^{\frac{1}{k}\cdot p} \sim M_{p,q}(1,1).$$

because in the case that $a = 1$ and $r = 1$, the number of arcs θ is ~ 1, and, also, the average over sub-balls disappears. (This holds for any $q \ge 1$ because of the locally constant property of the f_i: each f_i is locally constant on balls of radius 1. When restricted to functions that are locally constant on balls of radius 1, all L^q norms are comparable.) Meanwhile,

$$\begin{aligned} M_{p,p}(R,A) &= \prod_{i=1}^{k} \left(\sum_{\theta \ A^{-1}\text{-arcs}} \|f_{i,\theta}\|^2_{L^p(B_R)} \right)^{\frac{1}{2}\cdot\frac{1}{k}\cdot p} \\ &= \left(\prod_{i=1}^{k} \sum_{\theta \ A^{-1}\text{-arcs}} \|f_{i,\theta}\|^2_{L^p(B_R)} \right)^{\frac{1}{2}\cdot\frac{1}{k}\cdot p} \end{aligned} \tag{2}$$

The $p/2$ root of $M_{p,p}(R,A)$ is the geometric mean of the expression

$$\sum_{\theta\ A^{-1}\text{-arcs}} \|f_{i,\theta}\|^2_{L^p(B_R)}$$

The $p/2$ root of the RHS of (1) is the sum over i of the same, and is therefore at least as big. Thus, to prove (1) it suffices to prove that

$$M_{p,q}(1,1) \lesssim A^{\varepsilon} M_{p,p}(R,A) \tag{3}$$

for some q.

This will get harder as p increases. We begin with the case that $p = 2k$.

<u>$p = 2k$ case</u>

Supposing that $p = 2k$, we will prove (3) for $q = 2$. Starting with $r = 1$ and $a = 1$, we would like to increase r up to R and a up to A while multiplying $M_{p,q}(r,a)$ by a factor $\lesssim A^{\varepsilon}$. Each of the properties (O), (MKS1), and (MKS2) allows us to increase either r or a by squaring one of the parameters, which suggests that we might be able to prove (3) by repeatedly applying these moves in an appropriate sequence. None of these moves do anything if we start at $r = a = 1$. However, if we choose $r_0 > 1$ to be very small relative to A, say $r_0 = A^{\delta}$, then we have

$$M_{2k,2}(1,1), \lesssim r_o^{O(1)} K_{2k,2}(r_0,r_0).$$

Applying (MKS1) and then (O) gives

$$\begin{aligned} M_{2k,2}(r_0,r_0) &\overset{(MKS1)}{\lesssim} M_{2k,2}(r_0^2, r) \\ &\overset{(O)}{\lesssim} M_{2k,2}(r_0^2, r_0^2). \end{aligned}$$

Iterating this procedure gives

$$M_{2k,2}(r_0,r_0) \lesssim M_{2k,2}(A,A).$$

From here, we apply Hölder to give

$$M_{2k,2}(r_0,r_0) \lesssim M_{2k,2k}(A,A). \tag{4}$$

However, we want to show that

$$M_{2k,2}(r_0,r_0) \lesssim M_{2k,2k}(R,A).$$

To this end, we will use the following lemma, which says that $M_{p,p}$ is (weakly) monotonically increasing in r. The proof is similar in spirit to our proof of parallel decoupling.

Lemma 0.4. *Given $p, a, r_1,$ and r_2, with $r_1 \le r_2$, we have*

$$M_{p,p}(r_1, a) \le M_{p,p}(r_2, a).$$

Proof. Since $r_2 \le r_1$, we have that

$$M_{p,p}(r_1, a) = \underset{B_{r_2} \subset B_R}{Avg} \left[\underset{B_{r_1} \subset B_{r_2}}{Avg} \prod_i \left(\sum_\theta \|f_{i,m\theta}\|^2_{L^p(B_{r_1})} \right)^{\frac{1}{2} \cdot \frac{1}{k} \cdot p} \right]$$

The expression in brackets satisfies

$$\underset{B_{r_1} \subset B_{r_2}}{Avg} \prod_i \left(\sum_\theta \|f_{i,\theta}\|^2_{L^p(B_{r_1})} \right)^{\frac{1}{2} \cdot \frac{1}{k} \cdot p} \underset{\text{(Hölder)}}{\lesssim} \prod_i \left(\underset{B_{r_1} \subset B_{r_2}}{Avg} \left(\sum_\theta \|f_{i,\theta}\|^2_{L^p(B_{r_1})} \right)^{\frac{1}{2} \cdot p} \right)^{\frac{1}{k}}$$

$$\underset{\text{(Minkowski)}}{\lesssim} \prod_i \left(\sum_\theta \|f_{i,\theta}\|^2_{L^p(B_{r_1})} \right)^{\frac{1}{2} \cdot \frac{1}{k} \cdot p}.$$

To elaborate on the Minkowski step, note that

$$\underset{B_{r_1} \subset B_{r_2}}{Avg} \left((\cdot)^{p/2} \right) = \|\cdot\|_{l^{p/2}(B_{r_2})},$$

where

$$\| \cdot \|_{l^{p/2}(B_{r_2})}$$

is defined by a sum over balls B_{r_1} contained in B_{r_2}. We apply Minkowski's inequality in $l^{p/2}$ to the sum over θ to give

$$\begin{aligned} \underset{B_{r_1} \subset B_{r_2}}{Avg} \left(\sum_\theta \|f_{i,\theta}\|^2_{L^p(B_{r_1})} \right)^{p/2} &= \left\| \sum_\theta \|f_{i,\theta}\|^2_{L^p(B_{r_1})} \right\|^{p/2}_{l^{p/2}(B_{r_2})} \\ &\le \left[\sum_\theta \left\| \|f_{i,\theta}\|^2_{L^p(B_{r_1})} \right\|_{l^{p/2}(B_{r_2})} \right]^{p/2} \\ &= \left[\sum_\theta \|f_{i,\theta}\|^2_{L^p(B_{r_2})} \right]^{p/2}. \end{aligned}$$

□

Having proved Lemma 0.4, we resume from (4) to give

$$M_{2k,2}(r_0, r_0) \lesssim M_{2k,2k}(R, A).$$

This completes our proof of (3) in the case that $p = 2k$.

<u>$p > 2k$ case</u>

Supposing that $p > 2k$, we will prove (4) for $q = 2$. Again, if $r_0 = A^\delta$ for a sufficiently small δ (yet to be chosen), we have that

$$M_{p,2}(1,1) \lesssim r_0^{O(1)} M_{p,2}(r_0, r_0).$$

We use Hölder to bring 2 up to p/k so that we can apply (MKS2). This gives

$$\begin{aligned} M_{p,2}(r_0,r_0) &\le M_{p,\frac{p}{k}}(r_0,r_0) \\ &\overset{\text{(MKS2)}}{\lesssim} M_{p,\frac{p}{k}}(r_0^2,r_0) \\ &\overset{\text{(H)}}{\le} M_{p,2}(r_0^2,r_0)^\alpha\, M_{p,p}(r_0^2,r_0)^{1-\alpha} \\ &=: I^\alpha + II^{1-\alpha}, \end{aligned} \tag{5}$$

where α is chosen to satisfy $\frac{1}{p/k} = \frac{\alpha}{2} + \frac{1-\alpha}{p}$. We analyze I and II separately. Starting with I, we apply (O) and then Hölder to give

$$\begin{aligned} I = M_{p,2}(r_0^2,r_0) &\lesssim M_{p,2}(r_0^2,r_0^2) \\ &\le M_{p,p}(r_0^4,r_0^2)^{\alpha(1-\alpha)}\, M_{p,2}(r_0^4,r_0^2)^{\alpha^2}. \end{aligned} \tag{6}$$

Substituting this into (5) gives

$$M_{p,2}(r_0,r_0) \lesssim M_{p,p}(r_0^2,r_0)^{1-\alpha}\, M_{p,p}(r_0^4,r_0^2)^{\alpha(1-\alpha)}\, M_{p,2}(r_0^4,r_0^2)^{\alpha^2}.$$

We will repeat the sequence of moves from (6) many times. Supposing that $A = r_0^s$, we have

$$\begin{aligned} M_{p,2}(r_0,r_0) &\lesssim M_{p,p}(r_0^2,r_0)^{1-\alpha}\, M_{p,p}(r_0^4,r_0^2)^{\alpha(1-\alpha)}\ M_{p,p}(r_0^8,r_0^4)^{\alpha^2(1-\alpha)} M_{p,2}(r_0^8,r_0^4)^{\alpha^4} \\ &\lesssim \cdots \lesssim M_{p,p}(r_0^2,r_0)^{1-\alpha}\, M_{p,p}(r_0^4,r_0^2)^{\alpha(1-\alpha)}\ M_{p,p}(r_0^8,r_0^4)^{\alpha^2(1-\alpha)} \\ &\qquad \ldots M_{p,p}(A,A^{1/2})^{\alpha^{s-2}(1-\alpha)}\, M_{p,2}(A,A^{1/2})^{\alpha^{s-1}} \end{aligned}$$

Aside from the $M_{p,2}(A,A^{1/2})^{\alpha^{s-1}}$ factor, there are $s-1$ terms on the RHS. Let

$$III = M_{p,p}(r_0^2,r_0)^{1-\alpha}\, M_{p,p}(r_0^4,r_0^2)^{\alpha(1-\alpha)} \ldots M_{p,p}(A,A^{1/2})^{\alpha^{s-2}(1-\alpha)}.$$

After applying orthogonality one final time, we use the the fact that $M_{p,q}$ is increasing in q and then apply Lemma 0.4 to give

$$\begin{aligned} M_{p,2}(r_0,r_0) &\lesssim III \cdot M_{p,2}(A,A^{1/2})^{\alpha^{s-1}} \\ &\lesssim III \cdot M_{p,p}(A,A)^{\alpha^{s-1}} \\ &\lesssim III \cdot M_{p,p}(R,A)^{\alpha^{s-1}}. \end{aligned}$$

Since s is large, the contribution of $M_{p,p}(R,A)^{\alpha^{s-1}}$ to the RHS is very small.

To analyze III, we need a good way to estimate $M_{p,p}(r,a)$. For this, we will use induction on scales. Given $r \leq R$ and $a \leq A$, we have that

$$M_{p,p}(r,a) \lesssim M_{p,p}(R,a) = \prod_{i=1}^{k} \left(\sum_{\theta \ a^{-1}\text{-arcs}} \|f_{i,\theta}\|_{L^p(B_R)} \right)^{\frac{1}{2}\cdot\frac{1}{k}\cdot p}.$$

Each a^{-1} arc looks like the moment curve after an appropriate change of variables. We cut each a^{-1} inverse arc into A^{-1} arcs. This gives

$$\begin{aligned} M_{p,p}(r,a) &\lesssim D_{k,p}\left(\frac{A}{a}\right) \prod_{i=1}^{k} \left(\sum_{\theta \ A^{-1}\text{-arcs}} \|f_{i,\theta}\|_{L^p(B_R)} \right)^{\frac{1}{2}\cdot\frac{1}{k}\cdot p} \\ &= D_{k,p}\left(\frac{A}{a}\right) M_{p,p}(R,A). \end{aligned}$$

When we induct, we get that

$$III \lesssim D_{k,p}(A^{1-\delta})^{p(1-\alpha)} D_{k,p}(A^{1-2\delta})^{p\alpha(1-\alpha)} \dots D_{k,p}(A^{1/2})^{p\alpha^{s-2}(1\alpha)} M_{p,p}(R,A).$$

Recall that we wanted to prove that

$$M_{p,2}(1,1) \lesssim A^{\varepsilon} M_{p,p}(R,A).$$

So far, we have proved that an estimate of the form

$$M_{p,2}(1,1) \lesssim A^{C\delta} D_{k,p}(A^{1-2\delta})^{p\alpha(1-\alpha)} \dots D_{k,p}(A^{1/2})^{p\alpha^{s-2}(1-\alpha)} M_{p,p}(R,A).$$

We note that δ is yet to be chosen. We can choose it depending on ε. We have essentially proved that for small enough δ, we have

$$D_{k,p}(A) \lesssim D_{k,p}(A^{1-2\delta})^{p\alpha(1-\alpha)} \dots D_{k,p}(A^{1/2})^{p\alpha^{s-2}(1\alpha)}.$$

We want to show that $D_{k,p}(A) \lesssim A^{\varepsilon}$. For this, we suppose as an inductive hypothesis that

$$D_{k,p}\left(A^{1-\delta}\right) \lesssim \left(A^{1-\delta}\right)^{\varepsilon}$$

and then expand out the RHS of the above recurrence. Some algebra shows that the induction closes if and only if $\alpha \geq 1/2$. Recalling that we defined α in our Hölder step, we deduce that $\alpha \geq 1/2$ if and only if $p \leq 4k-2$. This completes the proof of the non-sharp decoupling theorem.

One may ask why our argument gave the upper bound $p \leq 4k-2$ rather than the upper bound from the sharp decoupling theorem. We recall that for a^{-1} caps θ, the functions f_θ were locally constant on 'planks' of dimensions $a \times a^2 \times a^3 \times \cdots \times a^k$. However, when we applied this fact to prove the two versions of Multilinear Kakeya for Slabs, intersected balls of radius a^2 with planks of these dimensions. The intersection of an $a \times a^2 \times a^3 \times \cdots \times a^k$ plank with a ball of radius a^2 is a slab of dimensions $a \times a^2 \times \cdots \times a^2$. Thus, we are really only using the weaker statement that the functions f_θ are locally constant on planks of dimensions $a \times a^2 \times \cdots \times a^2$.