18.118 DECOUPLING
LECTURE 18

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1. Decoupling for moment curve

Let us consider the moment curve $\Gamma_k \subset \mathbb{R}^k$ is defined to be

$$(t, t^2, \cdots, t^k) \text{ for } t \in [0, 1]$$

We want to study the decoupling for this curve, i.e. we want to study the decoupling of

$$N(\Gamma_k) = \sqcup N(A^{-1}\text{-arcs})$$

where N denotes a suitable tubular neighborhood.

Let

$$D_{p,k}(A) := D_p(N_{A^{-k}}(\Gamma_k) = \sqcup N(A^{-1}\text{-arcs}))$$

Later we will discuss about the choice of tubular radius A^{-k} here. The main purpose of the following classes is to show the following decoupling theorem:

Theorem 1.1 (Bourgain-Demeter-Guth). *If $2 \leq p \leq k(k+1)$, then*

$$D_{k,p}(A) \lesssim A^{\epsilon}$$

There is an old version of this theorem which need a stronger bound:

Theorem 1.2 (Bourgain-Demeter). *If $2 \leq p \leq 4k-2$, then*

$$D_{k,p}(A) \lesssim A6\epsilon$$

As a corollary of the theorem, we can prove a conjecture by Vinogradov

Corollary 1.3. *If $f(x) = \sum_{a=1}^{A} e(x_k a^k + \cdots + x_1 a)$, then*

$$\|f\|_{L^p([0,1]^k)} \lesssim A^{\epsilon} A^{\frac{p}{2}}$$

for $2 \leq p \leq k(k+1)$

Our goal is to show that if support of $\hat{f}$ is in $N_{A^{-k}}(\Gamma_k)$, then

$$\|f\|_{L^p} \lesssim A^{\epsilon} (\sum_{\theta} \|f_\theta\|_{L^p}^2)^{\frac{1}{2}}$$

here $2 \leq p$, and we will discuss about the sharp bound next time.

The main idea is following the broad/narrow estimate we used previously for the decoupling problem of paraboloid. Before that we first study some property of moment curve.

1.1. **Locally constant property.** Suppose θ is an a^{-1}-arc of the moment curve. More precisely, let

$$\gamma(t) = (t, t^2, \cdots, t^k)$$

then we suppose θ is $\gamma([t_0, t_0 + \frac{1}{a}])$. By Taylor expansion, we have

$$\gamma(t) = \gamma(t_0) + \gamma'(t_0)\Delta t + \frac{1}{2}\gamma''(t_0)\Delta t^2 + \cdots + \frac{1}{k!}\gamma^{(k)}(t_0)\Delta t^k + O(\Delta t^{k+1})$$

Note Δt^j is roughly scale of $\frac{1}{a^j}$. Moreover, $\gamma'(t_0), \gamma''(t_0), \cdots, \gamma^{(k)}(t_0)$ are transverse. Hence the arc θ lies in a box of scale $\frac{1}{a} \times \frac{1}{a^2} \times \cdots \times \frac{1}{a^k}$.

Remark 1.4. *Here our moment curve is given by*

$$(t, t^2, \cdots, t^k)$$

What about other curves? for example

$$\gamma(t) = (t, t^4, t^1 00)$$

Actually the above analysis still holds if t is large, let's say $t \in [\frac{1}{2}, 1]$. So everything is the same.

So if f_θ is a function such that the Fourier transform has support in a tubular neighborhood of θ, $f_\theta \sim$ constant in the dual rectangle, which is

$$a \times a^2 \times \cdots \times a^k$$

So in order to study the decoupling of moment curve, we want to study the intersection of these rectangles in different directions. We need something in the spirit of multilinear Kakeya.

1.2. **Multilinear Kakeya for slabs.** In previous classes we discussed multilinear Kakeya problem for tubes. Here we want to discuss the multilinear Kakeya for a new types of geometric objects: slabs. Slab is the tubular neighborhood of a hyperplane, and we will call the twice of the radius of this tubular neighborhood to be the thickness of the slab.

Let $S_{i,j}$ are characteristic functions of slabs of thickness 1 which are almost parallel to hyperplane $x_i^\perp$. Let $g_i = \sum_j w_{i,j} S_{i,j}$ for some constant $w_j \geq 0$.

Proposition 1.5.

$$\fint_{Q_s} \prod_{i=1}^k g_i \lesssim \prod_{i=1}^k \fint_{Q_s} g_i$$

Proof. Actually we can show $\lesssim$ is $\sim$.

First observe that for $S_1, \cdots, S_k$ be slabs in different direction,

$$|S_1 \cap S_2 \cap \cdots \cap S_k| \sim 1$$

Hence we have

$$\int_{Q_s} \prod g_i = \sum_{j_1, \cdots, j_k} w_{j_1} w_{j_2} \cdots w_{j_k} \int S_{1,j_1} S_{2,j_2} \cdots S_{k,j_k} \sim \prod_i (\sum_j w_{i,j})$$

By averaging we get

$$⨍_{Q_s} \prod_{i=1}^{k} g_i \sim \prod_{i=1}^{k} ⨍_{Q_s} g_i$$

□

Later this multilinear Kakeya for slabs will replace classical multilinear Kakeya in the proof of decoupling for moment curves.

1.3. **Analysis of width of tubular neighborhood.** In the beginning of this section, we define the decoupling constant $D_{k,p}(A)$ for moment curve to be the decoupling constant of the tubular neighborhood

$$D_p(N_{A^{-k}}(\Gamma) = \sqcup A^{-1}\text{-caps})$$

Here we clarify why the width of tubular neighborhood is chosen to be A^{-k}.

Recall in previous decoupling problem of paraboloid, when we do narrow estimate we need to analyze the decoupling in different scales. In moment curve case, we want to do similar estimate, hence we also need to study the moment curve in different scales.

Suppose τ is a K^{-1}-arc of Γ_k, and θ is an A^{-1}-arc of τ. Here we assume $A >> K$. Now let us apply a linear change of variable, which maps τ to be Γ_k and θ to be a $(\frac{A}{K})^{-1}$-arc of Γ_k.

For example, if τ is $\gamma([0, \frac{1}{K}])$, then the linear change of coordinate can be chosen to be

$$(w_1, w_2, \cdots, w_k) \to (Kw_1, K^2 w_2, \cdots, K^k w_k)$$

So the linear change of coordinate would change the scale at most $\sim K^k$. As a result under the linear change of coordinates W-neighborhood of τ would lie in $K^k W$-neighborhood of Γ_k.

Assume $W = W(A) = A^{-\sigma}$ should be the width of tubular neighborhood. Then after change of coordinates, $A^{-\sigma}$-neighborhood of θ would lie in $K^kA^{-\sigma}$ tubular neighborhood of $(\frac{A}{K})^{-1}$-arc. If we want to get the right scale, we need

$$K^kA^{-\sigma} \geq W(\frac{A}{K}) = (\frac{A}{K})^{-\sigma}$$

Hence the right power $\sigma = k$. And this is the reason we pick A^{-k}-neighborhood at the beginning of this decoupling problem.

1.4. **Broad and Narrow estimate.** Now we can start proving the decoupling estimate (in non-sharp case). We will follow the broad/narrow estimate we did for decoupling of paraboloid.

Let $\Gamma_k = \sqcup\tau$ which are K^{-1}-arcs, here $K << A$. Let $f = \sum_\tau f_\tau$. Define

$$S(x) = \{\tau : |f_\tau(x)| \geq \frac{1}{100(\sharp\tau)}|f(x)|\}$$

which are those contribute most of $|f(x)|$.

We say x is **broad** if there exists $\tau_1, \cdots, \tau_k \in S(x)$, such that τ_i's are well-separated, ; and the directions of the dual rectangles of them transverse. Here we need an observation. For a ball with radius a^2, $\tau^* \cap B(a^2)$ is roughly a $a \times a^2 \times \cdots \times a^2$ slab. So we can say directions for these rectangles, which are the direction of the slabs.

We say x is **narrow** else.

Narrow estimate:

First we observe that x narrow leads to $|S(x)| \lesssim 1$. In fact, if we have too many elements in $S(x)$, then we could pick many of them such that they are separated away, and as a result, they will form a well-separated set such that x belongs to Broad set. So

$$\int_{\text{narrow}} |f|^p \lesssim \sum_\tau \int |f_\tau|^p$$

We want to bound each τ by induction. Let us follow the setting we discussed in the analysis of width of tubular neighborhood, we get

$$\int |f_\tau|^p \lesssim D_p(\frac{A}{K})^p \left(\sum_{\theta: A^{-1}\text{-arcs}} \|f_\theta\|^2_{L^p} \right)^{\frac{1}{2}}$$

This is a good estimate we want.

Broad estimate:

$$\int_{\text{broad}} |f|^p \lesssim \sum_{\tau_1,\cdots,\tau_k} \int \prod_{i=1}^k |f_{\tau_i}|^{\frac{1}{k}p}$$

The right hand side term is in the form of multilinear expression, and our goal is to show it is less or equal than

$$A^{\epsilon}\Big(\sum_{\theta: A^{-1}\text{-arcs}} \|f_\theta\|_{L^2_{avg}}\Big)^{\frac{1}{2}p}$$

Now we want to do the two scale analysis we did previously. Define

$$M_{p,q}(r,a) := Avg_{B_r \subset B_R} \prod_{i=1}^{k} \left(\sum_{\theta: A^{-1}\text{-arcs}} \|f_{i,\theta}\|_{L^q_{avg}(B_r)} \right)^{\frac{1}{2}\frac{1}{k}p}$$

When r, a very small, $M_{p,q}(r,a)$ is almost just the multilinear expression. When r, a very big, $M_{p,q}(r,a)$ is the local expression we can deal with. The main idea (follows previous ideas) is to build up some induction process to connect these extreme cases.

Tool brainstorm: Let us review the tools we used in multilinear Kakeya.

- Hölder inequality: If $q_1 \leq q_2$, $M_{p,q_1} \leq M_{p,q_2}$;
- Orthogonality: When $q = 2$, by using local orthogonality we get if $a \leq r$, $M_{p,2}(r,a) \lesssim M_{p,2}(r,r)$;
- Multilinear Kakeya: We want to use slab version of multilinear Kakeya.

Now we want to use multilinear Kakeya for slabs. First we need to figure out what are $S_{i,j}$, i.e. those slabs. Since $|f_{i,\theta}|$ is almost a constant on $a \times a^2 \times \cdots \times a^k$ rectangle, so on B_{a^2}, $|f_{i,\theta}|$ is almost a constant on an a-neighborhood of a hyperplane, which is a slab.

Proposition 1.6. *(1)* $M_{2k,2}(a,a) \lesssim M_{2k,2}(a^2,a)$
(2) If $p \geq 2k$, $M_{p,\frac{2}{k}}(a,a) \lesssim M_{p,\frac{p}{k}}(a^2,a)$