18.118 DECOUPLING
LECTURE 14

INSTRUCTOR: LARRY GUTH
TRANSCRIBED BY SAHANA VASUDEVAN

Today we will talk about some applications of decoupling.

The first application is Strichartz inequality on tori. Let $T^d = \mathbb{R}^d/\mathbb{Z}^d$. For a lattice $\Lambda \subset \mathbb{R}^d$, let $T_\Lambda = \mathbb{R}^d/\Lambda$. For instance, we may take the rectangle $[0, \lambda_1] \times ... \times [0, \lambda_d]$ and identify opposite edges.

Our goal is to study the initial value problem for Schrodinger's equation. That is, let $u(x,t) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$ (or $T^d \times \mathbb{R} \to \mathbb{C}$ or $T_\Lambda \times \mathbb{R} \to \mathbb{C}$), where $x \in \mathbb{R}^d$, T^d, or T_Λ and $t \in \mathbb{R}$. We want to study the solutions to the partial differential equation

$$\partial_t u = i\triangle u$$

with initial conditions $u(x,0) = u_0(x)$. The following theorem was proved in the 1970s.

Theorem 0.1 (Strichartz)**.** *Suppose $u(x,t) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$ is a solution to the Schrodinger equation above. Let $p = \frac{2(d+2)}{d}$. Then*

$$\|u\|_{L^p(\mathbb{R}^d\times\mathbb{R})} \lesssim \|u_0\|_{L^p(\mathbb{R}^d)}.$$

Today, as an application of decoupling we shall show an analogue for tori. Before stating the theorem precisely we need to make a definition. Given an initial condition $u_0 : T_\Lambda \to \mathbb{C}$, we may write

$$u_0(x) = \sum_{\omega\in\Lambda^*} a_\omega e^{2\pi i\omega\cdot x}.$$

We say that u_0 has frequency $\leq N$ if $a_\omega = 0$ for $|\omega| > N$. We may now state the theorem.

Theorem 0.2 (Bourgain, Demeter)**.** *Suppose $u(x,t) : T_\Lambda \times \mathbb{R} \to \mathbb{C}$ is a solution to the Schrodinger equation with initial conditions $u(x_0) = u_0(x)$ where u_0 has frequency $\leq N$. Let $p = \frac{2(d+2)}{d}$. Then*

$$\|u\|_{L^p(T_\Lambda\times[0,1])} \leq C(\epsilon)C(\Lambda)N^\epsilon\|u_0\|_{L^2(T_\Lambda)}.$$

Remarks. (1) $C(\Lambda)$ is a nice function on the space of lattices; it is continuous and so uniformly bounded on compact subsets of the space of lattices.

(2) Why do we restrict to $t \in [0,1]$? On any finite volume Riemannian manifold (not $\mathbb{R}^d$!), $\|u(\cdot,t)\|_{L^p(M)} \geq \|u(\cdot,t)\|_{L^2(M)} = \|u_0\|_{L^2(M)}$ (the L^2-norm is independent of t), hence $\|u\|_{L^p(M\times\mathbb{R})} = +\infty$.

(3) The value of p is sharp.

(4) Previously, this theorem was known for T^1 and T^2, and there were partial non-sharp results in other cases.

Proof of Theorem 0.2. Let

$$u_0(x) = \sum_{\omega\in\Lambda^*,|\omega|\leq N} a_\omega e^{2\pi i\omega\cdot x}.$$

Then

$$u(x,t) = \sum_{\omega\in\Lambda^*,|\omega|\leq N} a_\omega e^{2\pi i\omega\cdot x+|\omega|^2 t}.$$

Let Ω be the set of frequencies $\{(\omega,|\omega|^2) : \omega \in \Lambda^*, |\omega| \leq N\}$ which appear in our expansion of u. Notice that Ω is contained in the paraboloid defined by $\omega_{d+1} = \sum_{i=1}^d \omega_i^2$. We consider the following rescaling which moves Ω to a subset of the unit truncated paraboloid:

$$L(\omega_1,...,\omega_{d+1}) = \left(\frac{\omega_1}{N},...,\frac{\omega_{d+1}}{N^2}\right).$$

Notice that $L(\Omega)$ is an N^{-1}-separated set of points on the unit truncated paraboloid. Let

$$\widetilde{u}(x,t) = \sum_{\omega\in\Lambda^*,|\omega|\leq N} a_\omega e^{2\pi i\left(\frac{\omega_1}{N}x_1+...+\frac{\omega_d}{N}x_d+\frac{|\omega|^2}{N^2}t\right)}.$$

By local decoupling (on $B^d_{N^2}\times B^1_{N^2}$), we have

$$\begin{aligned}\|\widetilde{u}\|_{Ł^p\left(B^d_{N^2}\times B^1_{N^2}\right)} &\lesssim \sum_{\omega\in\Lambda^*,|\omega|\leq N} \|a_\omega e^{2\pi i\left(\frac{\omega_1}{N}x_1+...+\frac{\omega_d}{N}x_d+\frac{|\omega|^2}{N^2}t\right)}\|^2_{Ł^p\left(B^d_{N^2}\times B^1_{N^2}\right)} \\ &= N^2\left(\sum_{\omega\in\Lambda^*,|\omega|\leq N}|a_\omega|^2\right)^{1/2} \\ &\lesssim N^\epsilon\|u_0\|_{L^2(T_\Lambda)}.\end{aligned}$$

Now note that if $(x,t)\in B^d_{N^2}\times B^1_{N^2}$ then $\left(\frac{x}{N},\frac{t}{N^2}\right)\in B^d_N\times B^1_1$. Thus

$$\|\widetilde{u}\|_{Ł^p(B^d_{N^2}\times B^1_{N^2})} = \|u\|_{Ł^p(B^1_N\times B^1_1)}.$$

The RHS is $\sim \|u\|_{Ł^p(T_\Lambda\times[0,1])} \sim \|u\|_{L^p(T_\Lambda\times[0,1])}$. This completes the proof. $\square$

Discussion of the proof of Strichartz. There are two key ingredients in the proof:

(1) L^2 conservation i.e. $\|u(\cdot,t)\|_{L^2(M)} = \|u_0\|_{L^2(M)}$ holds for any Riemannian manifold M (in particular $\mathbb{R}^d$, T^d and T_Λ).

(2) We have the dispersive estimate $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \lesssim t^{-d/2}\|u_0\|_{L^1(\mathbb{R}^d)}$ for $\mathbb{R}^d$.

The L^2-conservation and the dispersive estimate together can be used to prove Strichartz. The proof is clever but only uses Holder, duality, etc. In particular, whenever we have (1) and (2) Strichartz follows. In GR, people care about Strichartz (for the wave equation) on curved spacetime. The L^2-conservation is automatic, but people put a lot of work into proving the dispersive estimate.

The dispersive estimate is false on T^d. Take u_0 to be a frequency $\leq N$ approximately delta function with width $1/N$ with $\int u_0 = 1$. On $\mathbb{R}^d$, the dispersive estimate implies that if $|t| \sim 1$, then

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \lesssim t^{-d/2}\|u_0\|_{L^1(\mathbb{R}^d)} = 1.$$

On T^d, if $|t| \sim 1$,

$$\|u(\cdot,t)\|_{L^\infty(T^d)} \gtrsim \|u(\cdot,0)\|_{L^2(T^d)} = \|u_0\|_{L^2(T^d)} = N^{d/2}.$$

It can be even worse. On the unit cube torus, u is 1-periodic in t, which we can read off from the formula for u

$$u(x,t) = \sum_{\omega\in\mathbb{Z}^n, |\omega|\leq N} a_\omega e^{2\pi i(\omega x + |\omega|^2 t)}.$$

So

$$\|u(\cdot,1)\|_{L^\infty(T^d)} = \|u(\cdot,0)\|_{L^\infty(T^d)} \sim N^d$$

for the example above.

Proof sketch of dispersive estimate. Note that

$$\widehat{u}(\omega,t) = \widehat{u}_0(\omega)e^{i|\omega|^2 t},$$

hence

$$u(x,t) = \int_{\mathbb{R}^d} \widehat{u}_0(\omega)e^{i|\omega|^2 t}d\omega = S_t * u_0(x)$$

where $S_t(x) \approx t^{-d/2}e^{-i|x|^2/t}$. We have,

$$\begin{aligned}\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} &= \|S_t * u_0\|_{L^\infty(\mathbb{R}^d)} \\ &\leq \|S_t\|_{L^\infty(\mathbb{R}^d)}\|u_0\|_{L^1(\mathbb{R}^d)} \\ & t^{-d/2}\|u_0\|_{L^1(\mathbb{R}^d)}.\end{aligned}$$

as desired. ∎

On T^d, if

$$u_0(x) = \sum_{\omega \in \mathbb{Z}^n, |\omega| \leq N} a_\omega e^{2\pi i \omega x}$$

then

$$u(x,t) = \sum_{\omega \in \mathbb{Z}^n, |\omega| \leq N} a_\omega e^{2\pi i(\omega x + |\omega|^2 t)}.$$

Basic fact: for $\omega_1, \omega_2 \in \mathbb{Z}^n$, $e^{2\pi i \omega_1 x} *_{T^d} e^{2\pi i \omega_2 x} = 0$ if $\omega_1 \neq \omega_2$ and $e^{2\pi i \omega x}$ if $\omega_1 = \omega_2 = \omega$. So

$$u(x,t) = K_{n,t} * u_0(x)$$

where

$$K_{N,t}(x) = \sum_{\omega \in \mathbb{Z}^n, |\omega| \leq N} e^{2\pi i(\omega x + |\omega|^2 t)}.$$

Let us see what S_t and $K_{N,t}$ look like.

Picture 1. Picture of S_t. For a fixed t, S_t oscillates with uniform height, frequency of oscillation increases. As t increases, picture of S_t becomes shorter and wider, but still oscillating with uniform height and increasing frequency.

Note that $\|K_N\|_{L^2([0,1]^{d+1})} \sim N^{d/2}$. This suggests that $|K_N(x,t)| \sim N^{d/2}$ at a typical point (x,t). But K_N also has some much taller peaks. From the formula for K_N, we can read off that $K_N(0,1) = K_N(0,0) \sim N^d$. K_N also has tall peaks near rational points (with small denominators). For instance, consider the special case $d = 1$ and

$$K_N(0, 2/3) = \sum_{\omega \in \mathbb{Z}^d, |\omega| \leq N} e^{2\pi i (2/3)|\omega|^2} = \sum_{\omega=-N}^{N} e^{2\pi i (2/3)\omega^2}.$$

Notice that $e^{2\pi i (2/3)\omega^2}$ only depends on ω modulo 3. So if N is a multiple of 3, we get

$$K_N(0, 2/3) = \frac{N}{3} \sum_{\omega=0}^{2} e^{2\pi i (2/3)\omega^2}.$$

The sum on the right-hand side is not zero, and so $|K_N(0, 2/3)| \sim N$, and we see that K_N has another tall peak near $(0, 2/3)$.

Weyl's differencing method gives a good way of estimating $|K_N(x,t)|$. This method plays an important role in the circle method in analytic number theory, and we will discuss it more in the second half of the course (coming soon). Using Weyl's method, people have proven that $|K_N|$ has peaks only near to rational points of small denominator. In Bourgain's work on periodic Strichartz estimates in the early 90s, he was able to control the effect of these tall peaks using a mix of restriction theory and the circle method.

From the description so far, K_N already sounds a lot more complicated than the Schrodinger kernel S_t. But we haven't yet mentioned the most technically difficult aspect. While S_t oscillates in a regular way – with the speed of oscillation increasing smoothly – K_N oscillates in a chaotic way, with the speed of oscillation speeding up and slowing down in a jagged fashion. These oscillations presumably lead to a lot of cancellation, which would explain why the kernel K_N has almost the same L^p estimates as S_t, even though it is much larger. But actually proving that the chaotic-looking oscillations of K_t lead to cancellation was out of reach before decoupling.

Picture 2. Picture of K_N in terms of x and t. Big peaks near rational points (related to Weyl estimates and circle method), irregular oscillations.